CHARACTERISTIC CLASSES AND PROJECTIVE MODULES

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Ph. D. Thesis 1970

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Abstract.

This thesis is in 4 separate parts, of which Chapters 1 a nd 2 form the first part, and Chapters 3,4,5 are the remaining parts. The thesis is concerned with characteristic classes, which can be viewed as obstructions to stable triviality. In Chapter 1 we show just how much information the Chern classes carry in this respect, by proving that they determine the stable class upto a finite set. Chapter 3 discusses Chern classes of vector bundles over suspensions, and the effect on Chern classes of collapsing bundles. In Chapter 5 we introduce characteristic classes into algebraic K-theory and show that these have analogous properties to topological characteristic classes, in particular that the first characteristic class is a complete invariant for projective modules of rank 1. Chapter 2 concerns the topological process of blowing up a submanifold and the effect on characteristic classes. For this chapter we need to understand the theory of orientation, umkehr homomorphisms and Riemann-Roch theorems, and this is described in a fairly abstract sense in Chapter 1. Chapter 4 shows how the trace of an endomorphism (which in special cases can be considered as a first characteristic class) can be axiomatized, and proves the existence and uniqueness of a trace under certain finiteness conditions on the category.

Introduction.

This thesis is in 4 separate parts, each with its own introduction, references and page numbering. The first part is divided into 2 chapters. The chapters are as follows:

Chapter 1. Orientation.

Chapter 2. Blowing up submanifolds.

Chapter 3. Suspending and collapsing Chern classes.

Chapter 4. Trace in additive categories.

Chapter 5. Introduction to the theory of characteristic classes in algebraic K-theory.

The sections in each chapter are numbered, and the subsections are subnumbered. References to such numbers are to that chapter unless otherwise stated.

All theorems and propositions are original, (as are corollaries to such), unless otherwise stated either in the text, or in the relevant introduction. Lemmas are either technical, well-known, or both. The symbol ' ////. ' is used to denote the end or lack of proof.

There are many people I should like to thank for one thing or another in connection with this thesis, in particular:
Professor R.L.E.Schwarzenberger for supervising me, Professor J.Eells for encouraging me, and the Science Research Council for financing me. I should also like to record my gratitude to Professor I.M.James who taught me as an undergraduate, and who stimulated my interest in mathematics.

CHAPTER 1 : ORIENTATION

ORIENTATION (chapter 1)

and BLOWING UP CHERN CLASSES (Chapter 2)

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Chapter 1 concerns orientation of vector bundles and the umkehr homorphism. Given a cohomology theory, the definition of the umkehr homorphism for a certain class of embeddings of manifolds and its main properties (such as being adjoint to the induced homorphism) follow formally from the properties of a Thom isomorphism, or orientation for a certain class of vector bundles, namely we can define the umkehr homomorphism when the normal bundle is orientable. This is described in Dyer (19).

However, the question of whether of not a given vector bundle is orientabledoes not depend on formal properties of the cohomology theory, but in intimate relations between the type of vector bundle an. the theory. So the first approximation at formalising this situation is to require that along with the cohomology theory we are given a 'class' of vector bundles (obeying certain closure operations) which are orientable with respect to the theory.

But what does 'orientable' mean? According to Dyer (19), an n-plane bundle $E \longrightarrow X$ is orientable with respect to a cohomology theory h^* if there is an element $U \in h^n(E,E')$ (where E' is E minus the zero section) which restricts to a generator of $h^n(E_x,E_x') \cong h^n(S^n)$ for each $x \in X$. Using the axioms of cohomology (in particular the half-exactness) it then follows that

(A) $$.U : h^i(X) \longrightarrow h^{n+i}(E,E')$$

is an isomorphism for each i, ~~th~~ equivalently

(B) $h^*(E,E')$ is a free $h^*(X)$ module with one generator, namely U.

If we adopt (B) as the definition of orientability, then for our purposes we no longer need to know that h^* is half exact, or that it is graded, so we consider what we call multiplicative homotopy functors which are roughly speaking multiplicative half exact functors which are not necessarily half exact.

This is the approach of Chapter 1. There are 3 main advantages of this approach:

(i) the methods and results apply equally well to half exact functors as to cohomology theories

(ii) we do not have to worry about grading or suspension isomorphisms, although the classical 'graded' results for a cohomology theory are implied by our results bt taking $\hat{t} = \oplus h^*$).

(iii) the idea of having a class of vector bundles which are orientable allows us to look at vector bundles as possibly carrying some extra structure - such as admitting a complex structure - which we cannot do if we are only looking at orientation of n-plane bundles. In this sense, the approach is motivated by (16), where the umkehr homomorphism for K-theory is defined. It is this particular example which we need in Chapter 2.

In sections 1 and 2 we define a multiplicative homotopy functor and maps between them. This is modelled on the theory of half exact functors. In section 3, which was motivated by Grothendieck's work on λ-rings (21), we describe an detail a particular multiplicative homotopy functor, $\hat{G}$, which we call the Chern ring, and is in fact half exact. Prop.3.11 is a special version of a well known proposition ((18), Satz 8.9). Moreover, the Chern class defines a map $\hat{C}:\tilde{K} \longrightarrow \hat{G}$ and in section 4 we examine the relationship between thse two theories. By appealing to classioal results ((18), Satz 7.1) we deduce:

<u>4.4 Theorem</u> <u>Let X be a CW complex of dimension $\leqslant 5$. Then</u>

$$\hat{C}(X):K(X) \longrightarrow \hat{G}(X)$$

$$C(X):K(X) \longrightarrow G(X)$$

<u>are ring isomorphisms.</u>

This has some interesting corollaries (see 4.5, 4.6 and 4.8)

We also prove a general theorem on half exact functors which is interesting in its own right:

4.16 Theorem. Let W_*^f be the category of based finte CW complexes, let $t, t' : W_*^f \longrightarrow Ens_*$ be two half exact functors, and let $\rho : t \longrightarrow t'$ be a natural transformation. Then there exists a half exact functor $u : W_*^f \longrightarrow Ens_*$ and natural transformations $t'\Sigma \rightarrow u$ and $u \longrightarrow t$ such that the sequence

$$\cdots \rightarrow t\Sigma^n \longrightarrow t'\Sigma^n \longrightarrow u\Sigma^n \longrightarrow t\Sigma^{n-1} \cdots \rightarrow u \rightarrow t \xrightarrow[\rho]{} t'$$

is exact.

This construction is natural and moreover if t, t' take values in Ab then so does u.

(Remark: We conjecture that this u is unique upto natural equivalence.)

From a corollary of this (4.21) we deduce (4.22) that if X is a finite CW complex, then the chern classes of a vector bundle determine the stable class upto a finite set.

In section 5 we give some examples of the two preceeding sections and in section 6 show how Theorem 4.4 links up with the Atiyah-Hirzebruch spectral sequence. From section 7 on, we return to the general theory of multiplicative homotopy functors and orientation. Section 7 is mainly expository, except that we introduce relative Thom spaces and prove the relative analogues of well known results. Sections 8 to 11 are interpretations of similar results of Dyer (19) into the theory of multiplicative homotopy functors..In section 10 we prove a generalised Riemann-Roch theorem, and examine a particular case which we need in Chapter 2, namely that of $\hat{C} : K —— \hat{G}$. (Although this (10.6) would appear to be the Riemann-Roch theorem of Atiyah-Hirzebruch (9), ~~but as~~ we illustrate by an example (10.7) there is a minor error in this paper which gives the wrong equation.)

Corollary 10.4 is the well known result of (9).

Section 11 then introduces 'classes of vector bundles' and 'natural orientations' which enable us to define the umkehr homomorphism naturally. The most important example of this is for K-theory when we take the class of all complex vector bundles. This is used in Chapter 2.

Chapter 2 concerns topological process of blowing up a submanifold Y of a manifold X, which we describe in terms of a universal property (Prop.2.4). The geometric construction may be defined as follows. Let U be a tubular neighbourhood of Y in X and let E be the normal bundle. We may identify the boundary of $X \setminus U$ with the sphere bundle SE, and glue onto the boundary the projective bundle PE be the map $SE \longrightarrow PE$. The resulting topological space X' admits a natural structure as a manifold, and contains the manifold Y' = PE as a submanifold of codimension 1. The main theorem of this chapter is:

3.1 Theorem Let X be a compact complex manifold, and $i:Y \longrightarrow X$ a compact complex submanifold. Let

$$\begin{tikzcd} Y' \arrow[r, "j"] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ Y \arrow[r, "i"'] & X \end{tikzcd}$$

be the result of blowing up Y. Then the equation

$$j_*(g^*E - L) = f^*TX - TX'$$

holds in K(X'), where E,L are the normal bundles of Y,Y' and

$$j_*:K(Y') \longrightarrow K(X')$$

is the umkehr homomorphism.

This is the topological version of an algebro-geometric theorem of Porteous (10),(see 3.2 for the relation between these two versions) and in this sense it, and the proof, are original. The proof is based on an interpretation of K-theory due to Atiyah (1) in terms of difference bundles, and on the Bott isomorphism which enables us to use the machinery of Chapter 1. The proof is straighforward manipulation of di-fference bundles and reduces to proving that splittings of 4 exact sequences can be chosen in a suitable way. Essentially we choose the splittings locally and glue them together using partitions of unity, but there are some technical difficulties to be overcome. Finally we invoke the Riemann-Roch Theorem of Chapter 1 to obtain formulae concerning the effect on Chern character and Chern classes of blowing up a submanifold, analogous to those of Porteous.

There is an appendix to Chapter 1 , pages 47-49 in which we describe briefly with only sketch proofs the attempt to obtain a real version of section 4.

1 Multiplicative homotopy functors

1.1 Let $\underline{C}$ be an admissible category (in the sense of Eilenberg and Steenrod, 1) of locally compact hausdorff spaces. Let $\underline{C}_c \subseteq \underline{C}$ be the subcategory of compact spaces. Let $\underline{C}_*$ and $\underline{C}_{c*}$ be the corresponding based categories, and finally let $\underline{C}^2$ be the category of pairs (X,A) of objects of $\underline{C}$, where $A \subseteq X$ is a closed subspace. We tacitly assume in this Chapter that any construction we make belongs to the relevant category.

1.2 In particular, if $X \in \mathrm{Ob}\ \underline{C}$, we write X^+ for the disjoint union of X with a point, and X^* for the one point compactification of X. Then we assume that $X^* \in \mathrm{Ob}.C_c$ and note that if $X \in \mathrm{Ob}\ \underline{C}_c$, we have $X^+ = X^*$.

1.3 Definition

A multiplicative homotopy functor (abbreviated MHF) on $\underline{C}_{c*}$ consists of a contravariant functor

1.3.1 $$\hat{t}:\underline{C}_{c*} \longrightarrow \underline{\text{Abelian groups}}$$

together with a natural transformation

1.3.2 $$\hat{m}(X,Y):\hat{t}(X) \otimes \hat{t}(Y) \longrightarrow \hat{t}(X \wedge Y)$$

satisfying the following 5 conditions:

1.3.3 If $f \simeq g:X \longrightarrow Y$, then $\hat{t}(f) = \hat{t}(g):\hat{t}(Y) \longrightarrow \hat{t}(X)$.

1.3.4 $\hat{m}$ is associative: if $X,Y,Z \in \mathrm{Ob}\ C_{c*}$ then the following diagram commutes:

$$\begin{tikzcd}
\hat{t}(X) \otimes \hat{t}(Y) \otimes \hat{t}(Z) \arrow[r, "\hat{m}(X,Y) \otimes 1"] \arrow[d, "1 \otimes \hat{m}(Y,Z)"'] & \hat{t}(X \wedge Y) \otimes \hat{t}(Z) \arrow[d, "\hat{m}(X \wedge Y,Z)"] \\
\hat{t}(X) \otimes \hat{t}(Y \wedge Z) \arrow[r, "\hat{m}(X,Y \wedge Z)"'] & \hat{t}(X \wedge Y \wedge Z)
\end{tikzcd}$$

1.3.5 $\hat{m}$ is commutative: if $X,Y \in \mathrm{Ob}\ C_{c*}$, let $\tau:X \wedge Y \longrightarrow Y \wedge X$ be the transposition map. Then the following diagram commutes:

$$\begin{array}{ccc} \hat{t}(X) \otimes \hat{t}(Y) & \xrightarrow{\hat{m}(X,Y)} & \hat{t}(X \wedge Y) \\ \downarrow & & \downarrow \tau^* \\ \hat{t}(Y) \otimes \hat{t}(X) & \xrightarrow[\hat{m}(Y,X)]{} & \hat{t}(Y \wedge X) \end{array}$$

1.3.6 From the previous conditions it follows that $\hat{t}(X)$ is a commutative ring, with multiplication defined by the composite

$$\hat{t}(X) \otimes \hat{t}(X) \xrightarrow[\hat{m}(X,X)]{} \hat{t}(X \wedge X) \xrightarrow[\hat{t}(d)]{} \hat{t}(X)$$

where $d: X \longrightarrow X \wedge X$ is the diagonal map.

We require that $\hat{t}(S^0)$ has an identity.

1.3.7 From the previous conditions it follows that $\hat{t}(X)$ is a $\hat{t}(S^0)$ module (in fact it is an algebra) with action defined by

$$\hat{t}(S^0) \times \hat{t}(X) \xrightarrow{\hat{m}(S^0,X)} \hat{t}(S^0 \wedge X) \xrightarrow{=} \hat{t}(X)$$

where we identify $S^0 \wedge X$ with X under the natural homeomorphism.
We require that $\hat{t}(X)$ is a unital $\hat{t}(S^0)$ module.

1.3.8 Remark: The circumflex is used to denote that these functors are similar to reduced cohomology theories, and we shall later introduce the corresponding 'unreduced' theory. However, we do not require that $\hat{t}(\text{point}) = 0$.

1.4 A map $\hat{\alpha}: (\hat{t}, \hat{m}) \longrightarrow (\hat{s}, \hat{n})$ between two MHF is a natural transformation $\hat{\alpha}: \hat{t} \longrightarrow \hat{s}$ such taat

1.4.1 $\hat{\alpha}$ commutes with multiplication: the diagram

$$\begin{array}{ccc} \hat{t}(X) \otimes \hat{t}(Y) & \xrightarrow{\hat{m}(X,Y)} & \hat{t}(X \wedge Y) \\ \hat{\alpha}(X) \otimes \hat{\alpha}(Y) \downarrow & & \downarrow \hat{\alpha}(X \wedge Y) \\ \hat{s}(X) \otimes \hat{s}(Y) & \xrightarrow[\hat{n}(X,Y)]{} & \hat{s}(X \wedge Y) \end{array}$$

1.4.2 The map $\hat{\alpha}(S^0):\hat{t}(S^0) \longrightarrow \hat{s}(S^0)$ preserves the identity.

1.5 If $(\hat{t},\hat{m})$ is a MHF, we write $t(+)$ for $\hat{t}(S^0)$, and call this the coefficient ring.

2. Examples of multiplicative homotopy functors

2.1 Reduced K-theory, $\tilde{K}$, with multiplication induced by exterior product of vector bundles.

2.2 If H^* is singular cohomology theory, and n is a fixed integer then

$$\bigoplus_k \tilde{H}^{2kn}(\ -\ )$$

with multiplication given by cup product is a MHF.

2.3 If R is a commutative ring with identity, there is the constant functor $\hat{R}$ defined by

$$\hat{R}(X) = R \quad \text{for all } X \in \mathrm{Ob}\ \underline{C}_{c*}$$

$$\hat{R}(f) = \text{identity} \quad \text{for } f:X \longrightarrow Y$$

$\hat{m}(X,Y):R \otimes R \longrightarrow R$ is multiplication.

2.4 If $(\hat{t},\hat{m})$ and $(\hat{s},\hat{n})$ are MHF, then so is $(\hat{t} \times \hat{s},\hat{m} \times \hat{n})$ where

$$(\hat{t} \times \hat{s})(X) = \hat{t}(X) \times \hat{s}(X) \quad \text{(direct product of abelian groups)}$$

and multiplication is given by

$$\begin{tikzcd}
(\hat{t} \times \hat{s})(X) \otimes (\hat{t} \times \hat{s})(Y) \arrow[rr, dashed] \arrow[d, equal] & & (\hat{t} \times \hat{s})(X \wedge Y) \\
(\hat{t}X \times \hat{s}X) \otimes (\hat{t}Y \times \hat{s}Y) \arrow[dr, dashed] & & (\hat{t}X \otimes \hat{t}Y) \times (\hat{s}X \otimes \hat{s}Y) \arrow[u, "\hat{m} \times \hat{n}"'] \\
& (\hat{t}X \otimes \hat{t}Y) \times (\hat{t}X \otimes \hat{s}Y) \times (\hat{s}X \otimes \hat{t}Y) \times (\hat{s}X \times \hat{s}Y) \arrow[ur] &
\end{tikzcd}$$

The projections $\hat{t} \times \hat{s} \longrightarrow \hat{t}$ and $\hat{t} \times \hat{s} \longrightarrow \hat{s}$ are maps of MHF, and $\hat{t} \times \hat{s}$ is the product of $\hat{t}$ and $\hat{s}$ in the category of MHF.

2.5 If $(\hat{t},\hat{m})$ and $(\hat{s},\hat{n})$ are MHF, then so is $(\hat{t} \otimes \hat{s}, \hat{m} \otimes \hat{n})$ where

$$(\hat{t} \otimes \hat{s})(X) = \hat{t}(X) \otimes \hat{s}(X) \quad \text{(tensor product of abelian groups)}$$

and multiplication is given by

$$
\begin{tikzcd}
(\hat{t} \otimes \hat{s})(X) \otimes (\hat{t} \otimes \hat{s})(Y) \arrow[rr, dashed] \arrow[d, "\cong"] & & (\hat{t} \otimes \hat{s})(X \wedge Y) \\
\hat{t}(X) \otimes \hat{s}(X) \otimes \hat{t}(Y) \otimes \hat{s}(Y) \arrow[d, "\cong"] & & \\
\hat{t}(X) \otimes \hat{t}(Y) \otimes \hat{s}(X) \otimes \hat{s}(Y) \arrow[rr, "\hat{m}(X,Y) \otimes \hat{n}(X,Y)"'] & & \hat{t}(X \wedge Y) \otimes \hat{s}(X \wedge Y) \arrow[uu, "="']
\end{tikzcd}
$$

In general $\hat{t} \otimes \hat{s}$ is not the sum (in the category of MHF) as there may be no map $\hat{t} \longrightarrow \hat{t} \otimes \hat{s}$ for example. On the other hand, if $\hat{t}(X)$ and $\hat{s}(X)$ are rings with unit for all X, then the maps $\hat{t} \longrightarrow \hat{t} \otimes \hat{s}$ and $\hat{s} \longrightarrow \hat{t} \otimes \hat{s}$ defined by

$$\hat{t}(X) \longrightarrow \hat{t}(X) \otimes \hat{s}(X) \qquad a \longmapsto a \otimes 1$$

$$\hat{s}(X) \longrightarrow \hat{t}(X) \otimes \hat{s}(X) \qquad b \longmapsto 1 \otimes b$$

are maps of MHF, and in this case, $\hat{t} \otimes \hat{s}$ is the sum in this category.

3. The Chern ring

3.1 In this section we construct from singular cohomology theory H^* a MHF written $\hat{G}$ which we call the Chern MHF which has the property that if X is a compact connected space then $\hat{G}(X)$ is the set of formal sums

$$1 + a_1 + a_2 + \dots \qquad a_n \in H^{2n}(X, \mathbb{Z})$$

and such that the association of the Chern class to a complex vector bundle defines a map of MHF

$$\hat{c} : \hat{K} \longrightarrow \hat{G}.$$

Bearing this in mind, we see that addition and multiplication in $\hat{G}(X)$ must correspond to the formulae for calculating Chern classes of Whitney sums and tensor products. Since we allow vector bundles to have different dimensions on different components, such formulae have to be defined on each component seperately.

3.2 For $X \in \mathrm{Ob}\, \underline{C}_c$ a connected space, let $\hat{G}(X)$ be the set of formal sums (or what we may call 'formal power series')

3.2.1 $1 + a_1 + a_2 + \dots + a_n + \dots \qquad (a_n \in H^{2n}(X,\mathbb{Z}))$

where for convenience we define $a_0 = 1$. If $a_i = 0$ for $i > N$, then we abbreviate this formal power series to the formal polynomial

3.2.2 $$1 + a_1 + \dots + a_N.$$

We emphasize that these are formal expressions and that we are merely defining $\hat{G}(X)$ to be the set

3.2.3 $$\prod_{i=1}^{\infty} H^{2i}(X,\mathbb{Z})$$

and so the '1' and '+' have no significance other than the fact that this notation is in a convenient form to enable us to express addition and multiplication.

Define addition in $\hat{G}(X)$ to be 'formal multiplication of power series', that is

3.3 $$(1 + a_1 + \dots) \oplus (1 + b_1 + \dots) = (1 + c_1 + \dots)$$

where

3.3.1 $$c_k = a_k + a_{k-1}b_1 + \dots + a_1b_{k-1} + b_k = \sum_{i+j=k} a_ib_j$$

We write ' $\oplus$ ' for addition (and later ' $\otimes$ 'for multiplication) to display the vector bundle motivation and to avoid confusing the two kinds of addition (one in $\hat{G}(X)$ and one in $H^*(X)$).

3.4 This defines the structure of an abelian group on the set $\hat{G}(X)$, with zero element the zero sequence, 1 , (i.e. $1 + 0 + 0 + \dots$) To see that each element has an inverse, suppose that $1 + a_1 + \dots$ is in $\hat{G}(X)$. Then write formally

$$(1 + a_1 + \dots) \oplus (1 + x_1 + \dots) = 1$$

where x_i is an indeterminate and belongs to $H^{2i}(X,\mathbb{Z})$. Expanding the left hand side by the above formula (3.3) and equating terms of the same dimension we obtain equations

$$a_1 + x_1 = 0$$
$$a_2 + a_1x_1 + x_2 = 0$$
$$a_3 + a_2x_1 + a_1x_2 + x_3 = 0$$
$$\dots\dots$$

One can solve these equations uniquely for x by induction, e.g. $x_1 = -a_1$ from the first equation substituted into the second equation gives $x_2 = -a_2 + a_1^2$ and we may substitute these values for x_1 and x_2 into the third equation to determine x_3. (We note that x_i is given by a universal polynomial in $a_1, a_2, \dots, a_i$) For example, the inverse of $(1 + a)$ where $a \in H^2(X, \mathbb{Z})$ is the element

$$1 - a + a^2 - a^3 + a^4 - \dots .$$

and this shows that if the spaces under consideration have cohomology in arbitrarily large dimensions then we have to take 'formal power series' rather than 'formal polynomials' in order to get inverses. (In fact we could fall half way between these cases and consider the subgroup of $\hat{G}(X)$ generated by the formal polynomials, that is the 'formal rational series' since certainly the Chern classes lie in this subgroup.)

3.5 Now suppose that $X, Y \in \mathrm{Ob}\ \underline{C}_c$ and are connected. A map $f: X \longrightarrow Y$ induces maps $f^*: H^{2i}(Y) \longrightarrow H^{2i}(X)$ and hence a function $f^*: \hat{G}(Y) \longrightarrow \hat{G}(X)$ namely

$$f^*(1 + a_1 + a_2 + \dots) = 1 + f^*a_1 + f^*a_2 + \dots$$

Since $f^*: H^*(Y) \longrightarrow H^*(X)$ is a ring homomorphism, $f^*: \hat{G}(Y) \longrightarrow \hat{G}(X)$ is an abelian group homomorphism, which we define to be $\hat{G}(f)$. Thus $\hat{G}$ is a functor on the full subcategory of $\underline{C}_c$ of connected spaces. Denote the constant functor to $\mathbb{Z}$ from this category by $\mathbb{Z}$, and let $G = \mathbb{Z} \oplus \hat{G}$, that is $G(X) = \mathbb{Z} \oplus \hat{G}(X)$ and if $f: X \longrightarrow Y$, then $G(f) = 1 \oplus \hat{G}(f)$. Then G is a functor on this subcategory. We extend G to $\underline{C}_c$ by taking the sum over the components: If $X \in \mathrm{Ob}\ \underline{C}_c$, let $\{X_i : i \in I\}$ be the set of components. Since each $X_i \in \mathrm{Ob}\ \underline{C}_c$ and is connecte , we may define

3.6 $$G(X) = \bigoplus_{i \in I} G(X_i)$$

If $f: X \longrightarrow Y$ in $\underline{C}_c$ and X has components $\{X_i : i \in I\}$ and Y has components $\{Y_j : j \in J\}$ then the image of X_i under f is contained in one component of Y, say $Y_{f(i)}$, that is to say f induces a function $f: I \longrightarrow J$. Let $f_i : X \longrightarrow Y_{f(i)}$ be the restriction of f, and define $G(f): G(Y) \longrightarrow G(X)$ to be the map induced by the maps

3.7 $$\{G(f_i): G(Y_{f(i)}) \longrightarrow G(X_i) : i \in I\}$$

Then $G: \underline{C}_c \longrightarrow$ Abelian groups is a functor, and clearly homotopic maps induce the same homomorphism.

3.8 If $(X, *) \in \mathrm{Ob}\ \underline{C}_{c*}$, let $i: * \longrightarrow X$ be the inclusion of the base point, and define

$$\hat{G}(X) = \mathrm{Ker}\ (i^*: G(X) \longrightarrow G(*))$$

(Since pt. $\in \mathrm{Ob}\ \underline{C}_c$ is connected we know from 3.2 that $\hat{G}(\text{pt.}) = 0$, so $G(\text{pt.}) = \mathbb{Z}$. If $X \in \mathrm{Ob}\ \underline{C}_c$ is connected and $*$ is any base point, then $i^*: G(X) \longrightarrow G(*)$ is just projection

$$\mathbb{Z} \oplus \hat{G}(X) \longrightarrow \mathbb{Z}$$

so that $\hat{G}(X)$ is well defined, whether we use 3.2 or 3.8.)

As a set, G(X) is just the product of the $H^{2i}(X, \mathbb{Z})$ for $i \geq 0$, and as a set $\hat{G}(X)$ is the product of $\tilde{H}^{2i}(X, \mathbb{Z})$ for $i \geq 0$.

3.9 We now specialise to the case when $\underline{C}$ is the category of CW-complexes in order to invoke certain results on half-exact functors. Let $\underline{W}$ be the category of CW-complexes. Then by the above we have functors

$$G: \underline{W}_c \longrightarrow \text{Abelian groups}$$
$$\hat{G}: \underline{W}_{c*} \longrightarrow \text{Abelian groups}$$

3.10 Proposition

$\hat{G}$ is half exact, i.e. if $A \subseteq X$ is a subcomplex, the sequence

$$\hat{G}(X/A) \xrightarrow[p^*]{} \hat{G}(X) \xrightarrow[i^*]{} \hat{G}(A)$$

is exact.

Proof Without loss of generality we may assume that X is connected.

Let $A_0,\dots,A_r$ be the components of A, with $*\in A_0$. Let $i_r : A_r \to A$ be the inclusion. Then the sequence

$$\hat{G}(X/A) \xrightarrow{p^*} \hat{G}(X) \xrightarrow{i^*} \hat{G}(A)$$

is precisely

$$\hat{G}(X/A) \longrightarrow \hat{G}(X) \longrightarrow \hat{G}(A_0) \oplus \Big[\bigoplus_{j=1}^{r} G(A_j)\Big]$$

Since $pi = 0$, we have $i^*p^* = 0$.

Conversely, suppose $(1 + x_1 + \dots) \in \hat{G}(X)$ maps to zero under i^*. Then for each j and each n, $i_j^*(x_n) = 0$, so there is an element $y_{n,j} \in H^{2n}(X/A_j)$ which maps onto the element x_n under p^*, since $H^*(-)$ is half exact. From the pushout

$$\begin{array}{ccc} X & \longrightarrow & X/A_0 \\ \downarrow & & \downarrow \\ X/A_1 & \longrightarrow & X/A_0 \cup A_1 \end{array}$$

and the Mayer-Vietoris condition, there is an element $z_{n,1} \in H^{2n}(X/A_0 \cup A_1)$ which maps onto $y_{n,0}$ and onto $y_{n,1}$ under the natural maps.

By inductive use of the Mayer Vietoris condition, there is an element $z_n \in H^{2n}(X/A)$ which maps onto the element $y_{n,j} \in H^{2n}(X/A_j)$ for each j.

By considering the commutative diagram

$$\begin{array}{ccccc} H^{2n}(X/A) & \longrightarrow & H^{2n}(X) & \longrightarrow & H^{2n}(A) \\ \downarrow & \nearrow & & & \\ \bigoplus_j H^{2n}(X/A_j) & & & & \end{array}$$

we see that z_n maps onto x_n under i^*. The element $(1 + z_1 + \dots)$ belongs to $\hat{G}(X/A)$ and maps ont0 $(1 + x_1 + \dots)$ under i^*, whence $\hat{G}$ is half exact.

////.

<u>3.11 Proposition</u> <u>The sequence</u>

$$0 \longrightarrow \hat{G}(X \wedge Y) \longrightarrow G(X \times Y) \longrightarrow G(X \vee Y) \longrightarrow 0$$

<u>is exact.</u>

<u>Proof</u>: $X \vee Y$ is a subcomplex of $X \times Y$, and the cofibre of the inclusion is $X \wedge Y$. From the Puppe sequence of the inclusion we obtain a long

exact sequence

<u>3.11.1</u> $\hat{G}(X \vee Y) \xleftarrow{i^*} \hat{G}(X \times Y) \xleftarrow{p^*} \hat{G}(X \wedge Y) \longleftarrow \hat{G}(S(X \vee Y)) \xleftarrow{(Si)^*} \hat{G}(S(X \times Y))$

where S denotes (reduced) suspension.

Since $\hat{G}$ is half-exact, the inclusions $i_X: X \longrightarrow X \vee Y$ and $i_Y: Y \longrightarrow X \vee Y$ induce an isomorphism

<u>3.11.2</u> $\hat{G}(X \vee Y) \longrightarrow \hat{G}(X) \oplus \hat{G}(Y)$

If $p_X: X \times Y \longrightarrow X$ and $p_Y: X \times Y \longrightarrow Y$ are the projections, then they induce a map

<u>3.11.3</u> $\hat{G}(X) \oplus \hat{G}(Y) \longrightarrow \hat{G}(X \times Y)$

Let $q: \hat{G}(X \vee Y) \longrightarrow \hat{G}(X \times Y)$ be the composite of these two maps (3.11.2 and 3.11.1). In order to prove that $i^*q = 1$, it suffices to show that the composite

$$\hat{G}(X) \oplus \hat{G}(Y) \longrightarrow \hat{G}(X \times Y) \xrightarrow[i^*]{} \hat{G}(X \vee Y) \longrightarrow \hat{G}(X) \oplus \hat{G}(Y)$$

is the identity, which it is since

$$p_X i i_X = 1_X, \quad p_X i i_Y = 0, \quad p_Y i i_X = 0, \quad p_Y i i_Y = 1_Y.$$

Thus i^* is surjective. By similar argument applied to the half-exact functor $\hat{G}S(-)$ we see that $(Si)^*$ is surjective, and so the sequence

<u>3.11.4</u> $0 \longrightarrow \hat{G}(X \wedge Y) \xrightarrow[p^*]{} \hat{G}(X \times Y) \xrightarrow[i^*]{} \hat{G}(X \vee Y) \longrightarrow 0$

is exact (and moreover splits.) If we add to this the exact sequence

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow[1]{} \mathbb{Z} \longrightarrow 0$$

we obtain the exact sequence

$$0 \longrightarrow \hat{G}(X \wedge Y) \xrightarrow[p^*]{} \mathbb{Z} \oplus \hat{G}(X \times Y) \xrightarrow[i^*]{} \mathbb{Z} \oplus \hat{G}(X \vee Y) \longrightarrow 0$$

which is precisely the sequence

$$0 \longrightarrow \hat{G}(X \wedge Y) \xrightarrow[p^*]{} G(X \times Y) \xrightarrow[i^*]{} G(X \vee Y) \longrightarrow 0$$

////.

The next step is to define a natural ring structure on $G(X)$, to use this to define a natural multiplication $G(X) \otimes G(Y) \longrightarrow G(X \times Y)$ and then invoke Proposition 3.11 to show that this induces a product

$\hat{G}(X) \otimes \hat{G}(Y) \longrightarrow \hat{G}(X \wedge Y)$. As before, we start on the specail case that $X \in \mathrm{Ob}\, \underline{C}_c$ is connected.

3.12 Proposition*

Let $X \in \mathrm{Ob}\, \underline{C}_c$ be connected. There is a unique multiplication $\hat{m}_X : \hat{G}(X) \otimes \hat{G}(X) \longrightarrow \hat{G}(X)$, denoted by ' $\otimes$ ' such that

(i) if $(1 + x_1 + \dots) \otimes (1 + y_1 + \dots) = (1 + z_1 + \dots)$ where $x_i, y_i, z_i \subset H^{2i}(X, \mathbb{Z})$, then there is for each i a polynomial $Q_i(X_1, \dots, X_i, Y_1, \dots, Y_i)$ in $2i$ indeterminates with integer coefficients such that

$$z_i = Q_i(x_1, \dots, x_i, y_1, \dots, y_i)$$

(ii) if $x, y \in H^2(X, \mathbb{Z})$ then

$$(1 + x) \otimes (1 + y) = (1 + x + y) - (1 + x) - (1 + y)$$

Moreover, this multiplication is natural in X.

Proof: By (i) in order to calculate the i-th term of

$$(1 + x_1 + \dots) \otimes (1 + y_1 + \dots)$$

we can assume without loss of generality that $x_n = 0$ and $y_n = 0$ for $n > i$. If we write formally

$$1 + x_1 + \dots + x_i = (1 + a_1) \oplus (1 + a_2) \oplus \dots \oplus (1 + a_i)$$

where we think of the a's in $H^2(X, \mathbb{Z})$. Then x_j is the j-th symmetric function of the $a_1, \dots, a_i$.

Similarly we formally factorise

$$1 + y_1 + \dots + y_i = (1 + b_1) \oplus (1 + b_2) \oplus \dots \oplus (1 + b_i).$$

Then formally

$$\begin{aligned}(1 + x_1 + \dots) \otimes (1 + y_1 + \dots) &= \\ &\left[(1 + a_1) \oplus \dots \oplus (1 + a_i)\right] \otimes \left[(1 + b_1) + \dots + (1 + b_i)\right] \\ &= \bigoplus_{j,k} \left[(1 + a_j) \otimes (1 + b_k)\right] \\ &= \bigoplus_{j,k} \left[(1 + a_j + b_k) - (1 + a_j) - (1 + b_k)\right] \quad \text{by (ii)}\end{aligned}$$

The i-th term of the right hand side (written out in the usual form)

* This is essentially due to Grothendieck.

is a certain symmetric function of the a_r and the b_s and so is a unique polynomial in the symmetric functions of the a_r and b_s, that is in the x_r and y_s.

Multiplication is clearly natural since the terms of the product are polynomials and since, if $f:X \longrightarrow Y$, $\hat{G}(f)$ is defined in terms of $f^*:H^*(Y) \longrightarrow H^*(X)$ which is a ring homomorphism and so preserves polynomials.

////.

<u>3.13</u> In general the ring $\hat{G}(X)$ does not have a unit, and the process of adjoining a unit defines the ring structure on $G(X)$. More precisely we define multiplication in $G(X)$ by

<u>3.13.1</u> $(n,x) \otimes (m,y) = (nm, ny + mx + x \otimes y)$

where $(n,x),(m,y) \in G(X) = \mathbb{Z} \oplus \hat{G}(X)$.

Now if $X \in \mathrm{Ob}\ \underline{C}_c$ has components $X_1,\dots,X_r$ we have by definition

$$G(X) = \bigoplus_i G(X_i) \quad \text{(direct product of abelian groups)}$$

We give $G(X)$ the ring structure to make this a ring isomorphism, where the right hand side is given the ring structure of the direct product of the rings $G(X_i)$. This defines a multiplication in $\hat{G}(X)$ for $X \in \mathrm{Ob}\ \underline{C}_*$ (which is an ideal since it is the kernel of the ring homomorphism $\hat{G}(X) \longrightarrow G(*)$.)

<u>3.14</u> If $X,Y \in \mathrm{Ob}\ \underline{C}_c$, there is a natural multiplication

$$m(X,Y):G(X) \otimes G(Y) \longrightarrow G(X \times Y)$$

which is defined to be the composite

$$G(X) \otimes G(Y) \xrightarrow{p_X^* \otimes p_Y^*} G(X \times Y) \otimes G(X \times Y) \xrightarrow{m_{X \times Y}} G(X \times Y).$$

<u>3.15 Lemma</u> <u>For</u> $X,Y \in \mathrm{Ob}\ \underline{W}_{c*}$, <u>the composite</u>

$$\hat{G}(X) \otimes \hat{G}(Y) \longrightarrow G(X) \otimes G(Y) \xrightarrow{m(X,Y)} G(X \times Y) \xrightarrow{i^*} G(X \vee Y)$$

<u>is zero.</u>

<u>Proof:</u> Since multiplication preserves the augmentation, the image of this map is contained in $\hat{G}(X \vee Y)$, which we may identify with

$\hat{G}(X) \otimes \hat{G}(Y)$ by means of the isomorphism (3.11.2)

$$\hat{G}(X \vee Y) \xrightarrow[\begin{pmatrix} i_X{}^* \\ i_Y{}^* \end{pmatrix}]{} \hat{G}(X) \oplus \hat{G}(Y)$$

and hence it suffices to show that the composite maps

$$\hat{G}(X) \otimes \hat{G}(Y) \longrightarrow \hat{G}(X \vee Y) \xrightarrow{i_X{}^*} \hat{G}(X)$$

$$\hat{G}(X) \otimes \hat{G}(Y) \longrightarrow \hat{G}(X \vee Y) \xrightarrow{i_Y{}^*} \hat{G}(Y)$$

are zero. But this follows from naturality of multiplication. Let $j_X : * \longrightarrow X$ be the inclusion of the base point. Then the diagram

$$\begin{tikzcd}
\hat{G}(X) \otimes \hat{G}(Y) \arrow[r] \arrow[d, "j_X{}^* \otimes 1"'] & \hat{G}(X \vee Y) \arrow[d, "(j_X \vee 1)^*"] \arrow[dr, "i_Y{}^*"] & \\
\hat{G}(*) \otimes \hat{G}(Y) \arrow[r] & \hat{G}(* \vee Y) \arrow[r] & \hat{G}(Y)
\end{tikzcd}$$

commutes, so the composite $\longrightarrow\searrow$ is zero (since $\hat{G}(\text{pt.}) = 0$). Similarly for X.

////.

<u>3.16</u> Consider the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] \\
 & & \hat{G}(X \wedge Y) \arrow[d] \\
\hat{G}(X) \otimes \hat{G}(Y) \arrow[r] & G(X) \otimes G(Y) \arrow[r, "m(X,Y)"'] & G(X \times Y) \arrow[d] \\
 & & G(X \vee Y) \arrow[d] \\
 & & 0
\end{tikzcd} \tag{3.16.1}$$

By Proposition 3.11, the column is exact, and so by Lemma 3.15 there is a unique map

$$\hat{m}(X,Y) : \hat{G}(X) \otimes \hat{G}(Y) \longrightarrow \hat{G}(X \wedge Y)$$

making the diagram 3.16.1 commute. This multiplication is clearly natural, associative and commutative (in the sense of 1.3.4 and 1.3.5).

3.17 Theorem

$(\hat{G},\hat{m})$ is a multiplicative homotopy functor

Proof: We have to check conditions 1.3.3 to 1.3.7, the first three of which have already been verified. It remains to check 1.3.6 and 1.3.7.

Since S^0 is not connected, we must calculate $\hat{G}(S^0)$ via $G(S^0)$. Let $S^0 = \{0,1\}$, and take 0 as base point. Then $G(S^0) = G(0) \oplus G(1)$ (direct product of rings) and so $\hat{G}(S^0) = \text{Ker } (G(0) \oplus G(1) \longrightarrow G(o))$ i.e. $\hat{G}(S^0) = G(1) = G(pt.)$ Now $\{1\}$ is connected, so

$$G(1) = \mathbb{Z} \oplus \hat{G}(1)$$
$$= \mathbb{Z}$$

and so $\hat{G}(S^0) = \mathbb{Z}$, so is a ring with unit, so 1.3.6 is satisfied.

Now suppose $X \in \text{Ob } \underline{C}_{c*}$. In order to check that $\hat{G}(X)$ is a unital $\hat{G}(S^0)$ module, it suffices to check that $G(X)$ is a unital $G(1)$ module under the action

$$G(1) \otimes G(X) \xrightarrow{m(pt.,X)} G(\ 1 \times X) \ = G(X)$$

and without loss of generality we assume X is connected.

But multiplication is the composite (3.14)

$$G(1) \otimes G(X) \xrightarrow{p_1{}^* \otimes p_X{}^*} G(\ 1 \times X) \otimes G(1 \times X) \xrightarrow{\ m\ } G(1 \times X)$$

The image of the identity in $G(1)$ in $G(1 \times X)$ is the element $(1,1)$

Let $(n, 1 + x_1 + \ldots)$ belong to $G(X)$. Then by 3.12 and 3.13,

$$(1,1) \otimes (n,1 + x_1 + \ldots) \ = (n,\ 1 + x_1 + \ldots)$$

since $1 = 1 + 0 + 0 + \ldots$ is the zero element of $\hat{G}(X)$.

Thus $\hat{G}(X)$ is a unital $\hat{G}(S^0)$ module, which completes the proof of the theorem.

////.

3.18 Suppose $X \in \text{Ob } \underline{C}_c$ has components $\{X_i : i \in I\}$. If E is a complex vector bundle over X, let E_i be the part over X_i, and let $r(E_i)$ be the fibre dimension of E_i. This is a well defined integer since X_i is connected. Let $c(E_i)$ be the total Chern class of E_i, that is

$$c(E_i) = 1 + c_1(E_i) + \ \cdots \ + c_{r(E_i)}(E_i)$$

where $c_j(E_i) \in H^{2j}(X_i, \mathbb{Z})$. This means that $c(E_i) \in \hat{G}(X_i)$. Define the complete Chern class of E to be that element of G(X) which corresponds under the isomorphism

$$G(X) = \bigoplus_{i \in I} G(X_i)$$

to the element with components $(r(E_i), c(E_i))$ in $G(X_i)$. We denote the complete Chern class of E be C(E).

3.19 Proposition

If E,F *are complex vector bundles over* X *then*

(i) $C(E \oplus F) = C(E) \oplus C(F)$

(ii) $C(E \otimes F) = C(E) \otimes C(F)$

If $f:Y \longrightarrow X$ *in* $\underline{C}_c$, *then*

(iii) $C(f^*E) = f^*C(E)$,

where f^* *on the right hand side denotes* $G(f):G(X) \longrightarrow G(Y)$.

Proof: It suffices to consider the case when X,Y are connected. The formulae are then the standard ones (See Hirzebruch,[4],p.64).

////.

3.20 By 3.19 (i) and (iii), then C induces a natural transformation

$$C:K \longrightarrow G$$

and by 3.19 (ii) this is multiplicative. By naturality this induces a natural transformation

$$\hat{C}:\tilde{K} \longrightarrow \hat{G}$$

3.20 Proposition

(i) $\hat{C}:\tilde{K} \longrightarrow \hat{G}$ *is a map of multiplicative homotopy functors.*

(ii) *If we consider* $\tilde{K}$ *and* $\hat{G}$ *when defined on the category* $\underline{W}_{c^*}$ *then* $\hat{C}:\tilde{K} \longrightarrow \hat{G}$ *is a map of half-exact functors.*

Proof: This is a restatement of Proposition 3.20.

////.

4. The relation between K-theory and the Chern ring

4.1 We first calculate $\hat{G}$ for spheres. We already know (Theorem 3.17) that $\hat{G}(S^0) = \mathbb{Z}$, and since $H^{ev}(S^{2n+1}) = 0$, we see that $\hat{G}(S^{2n+1}) = 0$, for $n \geqslant 0$.

Let $u \in H^{2n}(S^{2n}, \mathbb{Z})$ be a generator. Then the elements of $\hat{G}(S^{2n})$ can be put into bijective correspondence with the integers by the map

$$\mathbb{Z} \longrightarrow \hat{G}(S^{2n}) \qquad p \longmapsto 1 + pu$$

But
$$\begin{aligned}(1 + mu) \oplus (1 + pu) &= 1 + (m + p)u + mpu^2 \\ &= 1 + (m + p)u \quad \text{since } u^2 = 0\end{aligned}$$

so this function is an isomorphism (of abelian groups).

(Remark: This is not a ring isomorphism, since multiplication in $\hat{G}(S^{2n})$ is trivial - i.e. all products are zero. This is because multiplication is defined via the diagonal map $d: S^{2n} \longrightarrow S^{2n} \wedge S^{2n} = S^{4n}$ which is null homotopic.)

Putting these results together we obtain:

4.2 Proposition

$$\hat{G}(S^0) = \mathbb{Z}$$
$$\hat{G}(S^{2n+1}) = 0 \qquad n \geqslant 0$$
$$\hat{G}(S^{2n}) = \mathbb{Z} \qquad n \geqslant 1$$

////.

4.3 Thus $\hat{G}$ has the same coefficient groups as K-theory, but these two theories are not the same, since for example K-theory has period 2 but $\hat{G}$ does not. (See also 5.7) This shows that a half-exact functor is not determined just by its coefficient groups. However, we shall now show that for low dimensional spaces these two theories coincide.

4.4 Theorem

Let X be a CW complex with $\dim X \leq 5$. Then

$$\hat{C}(X):\tilde{K}(X) \longrightarrow \hat{G}(X)$$

and $$C(X):K(X) \longrightarrow G(X)$$

are ring isomorphisms.

Proof: For $n = 0,1,3,5$, $\hat{C}(S^n):\tilde{K}(S^n) \longrightarrow \hat{G}(S^n)$ is an isomorphism. For $n = 2m$, $\tilde{K}(S^{2m}) \cong \hat{G}(S^{2m}) \cong \mathbb{Z}$. A well known theorem of Bott the m-th Chern class of any complex bundle over S^{2m} is a multiple of $(m - 1)!$ and by a result of Borel-Hirzebruch, [3] every such multiple does arise. Thus

$$\hat{C}(S^{2m}):\tilde{K}(S^{2m}) \longrightarrow \hat{G}(S^{2m})$$

is multiplication by $(m - 1)!$ and so is an isomorphism for $m = 1,2$. Hence $\hat{C}(S^n):\tilde{K}(S^n) \longrightarrow \hat{G}(S^n)$ is an isomorphism for $n \leq 5$, and so

$$\hat{C}(X):\tilde{K}(X) \longrightarrow \hat{G}(X)$$

is an isomorphism for CW complexes of dimension less that 5.

////.

4.5 Corollary

If X is a CW complex of dimension ≤ 5, then $K(X)$ is completely determined by $H^2(X,\mathbb{Z})$, $H^4(X,\mathbb{Z})$ and the cup product

$$H^2(X,\mathbb{Z}) \otimes H^2(X,\mathbb{Z}) \longrightarrow H^4(X,\mathbb{Z}).$$

////.

4.6 Corollary

A vector bundle over a CW complex of dimension ≤ 5 is stably trivial iff its Chern classes vanish.

////.

4.7 If X is a CW complex and $\dim X \leq 4$, then $\dim SX \leq 5$, so the above theorem and its corollaries apply to SX. In particular, 4.5 implies that $K^1(X)$ is completely determined by $H^2(SX,\mathbb{Z})$, $H^4(SX,\mathbb{Z})$ and the cup product $H^2(SX,\mathbb{Z}) \otimes H^2(SX,\mathbb{Z}) \longrightarrow H^4(SX,\mathbb{Z})$. Since cup products vanish for suspensions, we obtain:

4.8 Corollary

Let X be a CW complex of dimension $\leqslant 4$. Then $K^1(X)$ is completely determined by $H^1(X)$ and $H^3(X)$. In fact

$$K^1(X) \cong H^1(X) \oplus H^3(X)$$

Proof: By 4.5,

$$\hat{C}(SX):\tilde{K}(SX) \longrightarrow \hat{G}(SX)$$

is an isomorphism. Now $\hat{G}(SX)$ consists of elements of the form

$$1 + a_1 + a_2 \qquad \text{where } a_1 \in H^1(X,\mathbb{Z}) \text{ and } a_2 \in H^3(X,\mathbb{Z})$$

and with addition given by

$$(1 + a_1 + a_2) \oplus (1 + b_1 + b_2) = (1 + (a_1 + b_1) + (a_2 + b_2))$$

(by definition of addition, and since cup products vanish)

and so the map $\hat{G}(SX) \longrightarrow H^1(X,\mathbb{Z}) \oplus H^3(X,\mathbb{Z})$ defined by

$$(1 + a_1 + a_2) \longmapsto (a_1, a_2)$$

is an isomorphism (of abelian groups).

////.

4.9 Corollary

Let A be a ring which is torsion free as an abelian group, and let

$$d(A) = \sup \{m: A \otimes \mathbb{Z}_{(n-1)!} = 0 \text{ for all } n \leqslant m\}$$

Then

$$\hat{C}(X) \otimes 1 : \tilde{K}(X) \otimes A \longrightarrow \hat{G}(X) \otimes A$$

is a ring isomorphism if X is a CW complex and $\dim X \leqslant 2d(A) + 1$.

Proof: Since A is torsion free and hence flat, the result of tensoring a half-exact functor with A is again a half-exact functor. Thus

4.9.1 $$\hat{C}(-) \otimes 1 : \tilde{K}(-) \otimes A \longrightarrow \hat{G}(-) \otimes A$$

is a map of half-axact functors which (as in 4.4) is an isomorphism for odd dimensional spheres and S^0. From the exact sequence

4.9.2 $$0 \longrightarrow \tilde{K}(S^{2m}) \xrightarrow[\hat{C}(S^{2m})]{} \hat{G}(S^{2m}) \longrightarrow \mathbb{Z}_{(m-1)!} \longrightarrow 0$$

we obtain the exact sequence

4.9.3 $$0 \longrightarrow \tilde{K}(S^{2m}) \otimes A \xrightarrow[\hat{C}(S^{2m}) \otimes 1]{} \hat{G}(S^{2m}) \otimes A \longrightarrow \mathbb{Z}_{(m-1)!} \otimes A \longrightarrow 0$$

But by the hypothesis on A, we know that $\mathbb{Z}_{(m-1)!} \otimes A = 0$ for $m \leqslant d(A)$. Thus

$$\hat{C}(S^{2m}) \otimes 1 : \tilde{K}(S^{2m}) \otimes A \dashrightarrow \hat{G}(S^{2m}) \otimes A$$

is an isomorphism for $m \leqslant d(A)$, and so the natural transformation $\hat{C}(-) \otimes 1$ is an isomorphism on spheres of dimension less than or equal to $2d(A) + 1$ and so is an isomorphism on CW complexes of dimension less than or equal to $2d(A) + 1$.

////.

<u>4.10</u> If p is a prime, let $Z_{(p)}$ denote the ring obtained from Z by localising at the prime ideal generated by p. We may identify $Z_{(p)}$ with the subring of the rationals consisting of elements of the form a/b where p does not divide b. Hence $Z_{(p)}$ is torsion free, and if B is an abelian group, then the kernel of the natural map

$$B \longrightarrow B \otimes Z_{(p)}$$

is the subgroup of the torsion subgroup of B consisting of thoee elements whose torsion is not a multiple of p, which we write

4.10.1 $$\mathrm{Tor}_{\text{not } p}(B) = \{ b \in B : \exists n \notin pZ \text{ with } n.b = 0. \}$$

<u>4.11 Corollary</u>

<u>Let</u> X <u>be a CW complex of</u> dim X $\leqslant 2m + 1$, <u>and let</u> p <u>be a prime with</u> $p \geqslant m - 1$. <u>Then there is an exact sequence</u>

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Tor}_{\text{not } p}(\tilde{K}(X)) & \longrightarrow & \tilde{K}(X) & \longrightarrow & \tilde{K}(X) \otimes Z_{(p)} \\ & & \downarrow & & \downarrow \hat{C}(X) & & \hat{C}(X) \otimes 1 \downarrow \cong \\ 0 & \longrightarrow & \mathrm{Tor}_{\text{not } p}(\hat{G}(X)) & \longrightarrow & \hat{G}(X) & \longrightarrow & \hat{G}(X) \otimes Z_{(p)} \end{array}$$

<u>Proof</u>: Follows from Cor. 4.9, since for $n \leqslant m - 1 \leqslant p$, $1/n!$ is a unit in $Z_{(p)}$ so if $x \in \mathbb{Z}_{n!}$,

$$x \otimes \frac{a}{b} = x \otimes \frac{n!}{1} \cdot \frac{a}{b.n!} = n!x \otimes \frac{a}{b.n!} = 0$$

////.

4.12 The previous corollary (4.11) shows how much information the Chern classes carry when we take into account torsion which is not a multiple of a given prime. The next corollary relates to torsion which is a multiple of certain primes.

If n is an integer (not necessarily prime), let A_n denote the subring of the rationals consisting of those fractions which can be written in the form a/b where $b = n^r$ for some r. Then A_n is torsion free, and if B is an abelian group, then the kernel of the natural map

$$B \longrightarrow B \otimes A_n$$

is the subgroup of torsion elements whose torsion is a power of n, which we write

4.12.1 $$\mathrm{Tor}_{(n)}(B) = \{ b \in B : \exists\, r \text{ with } n^r b = 0 \}$$

4.13 Corollary

Let X be a CW complex of dimension $\leq 2m + 1$. Let n (or n(m)) be the product of the primes less than or equal to m - 1. Then there is an exact sequence

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Tor}_{(n)}(\tilde{K}(X)) & \longrightarrow & \tilde{K}(X) & \longrightarrow & \tilde{K}(X) \otimes A_n \\
& \downarrow & & \downarrow \hat{c}(X) & & \cong \downarrow \hat{c}(X) \otimes 1 \\
0 \longrightarrow & \mathrm{Tor}_{(n)}(\hat{G}(X)) & \longrightarrow & \hat{G}(X) & \longrightarrow & \hat{G}(X) \otimes A_n
\end{array}$$

////

4.14 Roughly speaking, Cor.4.13 says that Chern classes classify stable bundles over 2m + 1 complexes up to n(m) torsion. Although n(m) tends to infinity with m, it does so much less rapidly than (m - 1)! and the jumps in its value take place less often as m increases. However it is probably the low dimensional cases to which these results will best apply. For example, for CW complexes of dimension less than or equal to 7 we are interested in 2-torsion, and for dimension less than or equal to 11 in 2- and 3-torsion.

<u>4.15</u> In order to obtain information on the relationship between K-theory and the Chern theory in general, we prove a general theorem which has many interesting applications.

Let $\underline{W}_*$ be the category of CW complexes, and $\underline{W}_*^f$ the subcategory of finite CW complexes. Let $\underline{Ens}_*$ be the category of based sets.

<u>4.16 Theorem</u>

<u>Let</u> $t,t':\underline{W}_*^f \longrightarrow \underline{Ens}_*$ <u>be two half exact functors with countable coefficient groups, and</u> $\rho:t \longrightarrow t'$ <u>a natural transformation. Then there exists a half exact functor</u> $u:\underline{W}_*^f \longrightarrow \underline{Ens}_*$ <u>and natural transformations</u> $t'\Sigma \longrightarrow u$ <u>and</u> $u \longrightarrow t$ <u>such that the sequence</u>

$$\rightarrow t\Sigma^n \longrightarrow t'\Sigma^n \longrightarrow u\Sigma^n \longrightarrow t\Sigma^{n-1} \rightarrow \cdots \longrightarrow u \rightarrow t \xrightarrow[\rho]{} t'$$

<u>is exact.</u>

<u>This construction is natural and moreover if</u> t,t' <u>take values in</u> <u>Ab then so does</u> u.

<u>Proof</u>: By Brown's Theorem (6), there are spaces $E,B \in \underline{W}_*$ such that on $\underline{W}_*^f$ the functors t,t' are naturally equivalent to $[-,E]$, $[-,B]$ respectively, and there exists a map $p:E \longrightarrow B$ such that the diagram

<u>4.16.1</u>
$$\begin{array}{ccc} t(X) & \longrightarrow & [X,E] \\ {\scriptstyle t(X)}\downarrow & & \downarrow{\scriptstyle p_*} \\ t'(X) & \longrightarrow & [X,B] \end{array}$$

commutes for $X \in \underline{W}_*^f$.

We can assume WLOG that p is a fibration. Let $i:F \longrightarrow E$ be the fibre over the base point. Define $u(X) = [X,F]$. Then the exact Puppe sequence

<u>4.16.2</u> $\rightarrow [X,\Omega E] \longrightarrow [X,\Omega B] \longrightarrow [X,F] \longrightarrow [X,E] \longrightarrow [X,B]$

becomes the required exact sequence on $\underline{W}_*^f$.

If t,t' take values in <u>Ab</u> then E,B are H-spaces and the map $p:E \longrightarrow B$ is an H-map. Under these conditions F is an H-space such that the inclusion $i:F \longrightarrow E$ is an H-map. To see this, consider the

4.16.3

$$\begin{tikzcd} F \times F \arrow[r, dashed] & E \times E \arrow[r, "m"] \arrow[d, "p\times p"'] & E \arrow[d, "r"] \\ & B \times B \arrow[r, "m"'] & B \end{tikzcd}$$

The square commutes upto homotopy, so the composite $F \times F \longrightarrow B$ is nul-homotopic, and hence there is a map $F \times F \longrightarrow F$ which defines the multiplication on F such that $i: F \longrightarrow E$ is an H-map.

////.

4.17 Examples

4.17.1 If $t = t' = H^m(-,\mathbb{Z})$ and ρ is multiplication by n, then $u = H^{m-1}(X,\mathbb{Z}_n)$ and the sequence is the usual one:

$$H^{m-1}(X) \xrightarrow{\times n} H^{m-1}(X) \longrightarrow H^{m-1}(X,\mathbb{Z}_n) \longrightarrow H^m(X) \xrightarrow{\times n} H^m(X)$$

4.17.2 If $H \hookrightarrow G$ are topological groups, there is a natural map (4)

$H^1(X,H_c) \longrightarrow H^1(X,G_c)$. In this case $u(X) = [X,G/H]$.

4.17.3 Complexification $\widetilde{KO} \longrightarrow \widetilde{K}$ and decomplexification $\widetilde{K} \longrightarrow \widetilde{KO}$ yield the usual exact sequences (7).

4.18 The case which we are interested is that of $\mathcal{E}: \mathcal{K} \longrightarrow \mathcal{G}$. Let u denote the corresponding half exact functor.

4.19 Proposition

$u(S^{2n}) = 0$ and $u(S^{2n+1}) = \mathbb{Z}_{n!}$

Proof. The exact sequence of Theorem 4.16 applied to S^{2n+1} becomes

$$\begin{array}{ccccccc} \mathcal{K}(S^{2n+2}) & \longrightarrow & \mathcal{G}(S^{2n+2}) & \longrightarrow & u(S^{2n+1}) & \longrightarrow & \mathcal{K}(S^{2n+1}) \\ \| & & \| & & & & \| \\ \mathbb{Z} & & \mathbb{Z} & & & & 0 \end{array}$$

so s $u(S^{2n+1}) = \mathbb{Z}_{n!}$ by Theorem 4.4. The result for S^{2n} follows similarly.

////.

4.19 It follows that if $X \in W^f_*$ then $u(X)$ is finite. In the following Proposition, we give a bound for the size of $u(X)$. Let $d(X)$ be the order of $u(X)$.

4.20 Proposition

Suppose X is a finite CW complex of dimension $2p+$ or $2p+2$. Let $n(r)$ be the number of r-cells. Then

$$d(X) = 1 \qquad \text{if } p = 0,1$$

$$d(X) \text{ divides } (2!)^{n(5)}(3!)^{n(7)} \cdots \quad (p!)^{n(2p+1)} \qquad p \geq 2$$

Proof. From Lagrange's Theorem, if $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is an e act sequence of finite groups, then the order of B is the product of the orders of A and C. If $A \longrightarrow B \xrightarrow{p} C$ is exact saquence of finite groups, let $K = \operatorname{Ker} p$. Then K is a quotient of A and B/K is a subgroup of C. This implies that the order of B divides the product of the orders of A and B.

We prove the theorem by induction on p. For $p = 0,1$ this follows from Theorem 4.4. Suppose the Proposition is proved for $p = 0,1,\ldots,q$.

4.20.1 Suppose $\dim X = 2q+1$. From the Puppe sequence

$$\bigvee_{n(2q+1)} S^{2q} \longrightarrow X^{(2q)} \longrightarrow X \longrightarrow \bigvee_{n(2q+1)} S^{2q+1}$$

we obtain the exact sequence

$$\begin{array}{ccccccc} \bigoplus_{n(2q+1)} u(S^{2q}) & \longleftarrow & u(X^{(2q)}) & \longleftarrow & u(X) & \longleftarrow & \bigoplus_{n(2q+1)} u(S^{2n+1}) \\ \| & & & & & & \| \\ 0 & & & & & & \bigoplus_{n(2q+1)} \mathbb{Z}_{q!} \end{array}$$

and so $d(X)$ divides $d(X^{(2q)}).(q!)^{n(2q+1)}$.

4.20.2 Suppose $\dim X = 2q+2$. From the Puppe sequence, we obtain the exact sequence

$$\bigoplus_{n(2q+2)} u(S^{2q+1}) \longleftarrow u(X^{(2q+1)}) \longleftarrow u(X) \longleftarrow 0$$

i.e. $u(X)$ is a subgroup of $u(X^{(2q+1)})$ so $d(X)$ divides $d(X^{(2q+1)})$.

This completes the induction.

////.

4.21 Corollary

$\mathrm{Ker}\left[\hat{C}(X):\hat{K}(X) \longrightarrow \hat{G}(X)\right]$ for X a finite CW complex is a finite group whose order divides

$$\prod_{p \geq 2} (p!)^{n(2p+1)}$$

where $n(r)$ is the number of r-cells. ////.

4.22 Corollary

For X a finite CW complex, the chern class of a vector bundle determines the stable class upto a finite set.

////.

4.23 Theorem

There is a natural transformation $e:\tilde{K} \longrightarrow \tilde{H}^{ev}(-;\mathbb{Z})$ such that the sequence

$$u(S^2X) \longrightarrow \tilde{K}(X) \longrightarrow \tilde{H}^{ev}(X) \longrightarrow u(SX) \longrightarrow K^1(X) \longrightarrow H^{odd}(X)$$

$$\longrightarrow u(X) \longrightarrow \tilde{K}(X) \longrightarrow \hat{G}(X)$$

is exact. In particular, for X a finite CW complex

$$e(X):\tilde{K}(X) \longrightarrow \tilde{H}^{ev}(X)$$
$$e(SX):K^1(X) \longrightarrow H^{odd}(X)$$

are isomorphisms modulo finite groups.

Proof. The exact sequence of Theorem 4.16 yields

$$u(S^2X) \to K(S^2X) \to \hat{G}(S^2X) \to u(SX) \to \hat{K}(SX) \to \hat{G}(SX) \to u(X) \to K(X) \to \hat{G}(X)$$

Since products in the cohomology of suspensions are zero, the natural function isomorphisms

$$\hat{G}(SX) \longrightarrow H^{odd}(X)$$
$$\hat{G}(S^2X) \longrightarrow H^{ev}(X)$$

are abelian group isomorphisms. Let $e:\tilde{K}(X) \longrightarrow \tilde{H}^{ev}(X)$ be the composite

$$K(X) \xrightarrow[\cong]{\text{Bott iso.}} K(S^2X) \xrightarrow[C(S^2X)]{} \hat{G}(S^2X) \xrightarrow{=} H^{ev}(X).$$

The exact sequence required then follows by substitution.

////.

4.24 By means of Prop.4.20, we can improve on Th.4.20 by giving some bounds for the size of the kernel and cokernel of e.

4.25 Corollary

Let X be a finite CW complex, and let $e(X):K(X) \longrightarrow H^{ev}(X)$ be the map described in Th.4.23. Then

$$|\operatorname{Ker} e(X)| \ \text{ divides } \prod_{p \geq 2} (p!)^{n(2p)}$$

$$|\operatorname{Coker} e(X)| \ \text{ divides } \prod_{p \geq 2} (p!)^{n(2p-1)}$$

where n(r) is the number of r-cells in X.

Proof. Apply Prop.4.20 and Th.4.23 and use the fact that r-cells in X correspond to r+1 cells in SX.

////.

Hence we obtain the analogues of Theorem 4.4 and Cor. 4.6.

4.19 Theorem

If X is a CW complex of dimension $\leqslant 2$, then

$$\hat{W}(X):KO(X) \longrightarrow \hat{G}O(X)$$

$$W(X):KO(X) \longrightarrow GO(X)$$

are isomorphisms.

////.

4.20 Corollary

A real vector bundle over a CW complex of dimension $\leqslant 2$ is stably trivial iff its Stiefel-Whitney classes vanish.

////.

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5. Examples and applications

5.1 We start this section by calculating $\hat{G}(P_n(R))$ for $n \leqslant 6$, where $P_n(R)$ denoted real projective n-space. Let $i:P_n(R) \longrightarrow P_{n+1}(R)$ be the inclusion. Since $\hat{G}$ is defined in terms of even cohomology, we see that

$$i^*:\hat{G}(P_{2n+1}(R)) \longrightarrow \hat{G}(P_{2n}(R))$$

is a ring isomorphism, so it suffices to calculate $\hat{G}(P_{2n}(R))$.

5.2 $\hat{G}(P_2(R))$

The only relevant cohomology group is $H^2(P_2(R),Z) = Z_2$ and $\hat{G}(P_2(R))$ has two elements, namely 1 and $1 + a$, where a is the non-zero element of $H^2(P_2(R))$, so additively $\hat{G}(P_2(R))$ is Z_2 generated by $1 + a$. The formula for multiplication (Theorem 3.12 (ii)) gives

$$\begin{aligned}(1 + a) \otimes (1 + a) &= (1 + a + a) - 2(1 + a)\\ &= 1 \quad \text{(the zero element)}\end{aligned}$$

so that multiplication is trivial (i.e. all products are zero).

5.3 $\hat{G}(P_4(R))$

$\hat{G}(P_4(R))$ consists of 4 elements, namely

$$1\ ,\ 1 + a,\ 1 + a^2,\ 1 + a + a^2.$$

and since $(1 + a) \oplus (1 + a) = (1 + a^2) \neq 0$ we see that $1 + a$ has order 4, and so generates the group, so that additively $\hat{G}(P_4(R)) = Z_4$. In order to determine the multiplication it suffices to calculate $(1 + a) \otimes (1 + a)$, which by the formula is

$$(1 + a + a) - 2(1 + a)$$
$$= 2(1 + a).$$

So $\hat{G}(P_4(R))$ is a cyclic group of order 4, generated by an element $x = (1 + a)$ and with multiplication described by $x^2 = 2x$.

5.4 $\hat{G}(P_6(R))$.

$\hat{G}(P_6(R))$ is a group of order 8 whose elements are $1,\ 1 + a,\ 1 + a^2,\ 1 + a^3, 1 + a + a^2, 1 + a + a^3,\ 1 + a^2 + a^3, 1 + a + a^2 + a^3$ where a generates $H^2(P_6(R))$. The inclusion

$$i : P_4(R) \longrightarrow P_6(R)$$

induces a surjection

$$i^* : \hat{G}(P_6(R)) \longrightarrow \hat{G}(P_4(R)$$

with kernel the two elements

$$1,\ 1 + a^3 \qquad (\text{i.e. } H^6(P_6(R), Z))$$

Thus we have a short exact sequence

5.4.1 $$0 \longrightarrow Z_2 \longrightarrow \hat{G}(P_6(R)) \longrightarrow Z_4 \longrightarrow 0$$

5.4.2 Lemma

<u>The element</u> $1 + a$ $\hat{G}(P_6(R))$ <u>has order 4</u>.

<u>Proof</u>: $2(1 + a) = 1 + a^2 \neq 0$

$4(1 + a) = 2(1 + a^2) = 0$ (since $a^4 = 0$)

////.

(Remark: Since $\hat{G}(P_n(R))$ is a group of order 2^n, every element has order a power of 2)

Let $q:Z_4 = \hat{G}(P_4(R)) \longrightarrow \hat{G}(P_6(R))$ be the map defined by

$$q(\,n(1 + a)) = n(1 + a)$$

where on the left hand side, a is the generator of $H^2(P_4(R),Z)$ but on the right hand side a is the generator of $H^2(P_6(R),Z)$. By Lemma Lemma 5.4.2, q is a homomorphism, and so the sequence 5.4.1 splits. Hence $\hat{G}(P_6(R) = Z_4 \oplus Z_2$ where $1 + a$ generates Z_4 and $1 + a^3$ generates Z_2. In order to determine the multiplication, it suffices to calculate the three products

$$(1 + a) \otimes (1 + a)$$
$$(1 + a) \otimes (1 + a^3)$$
$$(1 + a^3) \otimes (1 + a^3)$$

<u>5.4.3 Lemma</u>

(i) $(1 + a) \otimes (1 + a) = 2(1 + a)$

(ii) $(1 + a) \otimes (1 + a^3) = (1 + a^3) \otimes (1 + a^3) = 1$ (the zero element)

<u>Proof</u>. (i) follows as in 5.3 by virtue of 5.4.2

(ii) We formally factorise $1 + a^3 = (1 + x_1) \oplus (1 + x_2) \oplus (1 + x_3)$ so that $x_1 + x_2 + x_3 = 0$, $x_1x_2 + x_2x_3 + x_3x_1 = 0$, $x_1x_2x_3 = a^3$ where we think of x_i as having cohomological dimension 2.

Then

$$\begin{aligned}(1 + a) \otimes (1 + a^3) &= (1 + a) \times \left[\oplus (1 + x_i)\right] \\ &= \bigoplus_i \left[(1 + a) \otimes (1 + x_i)\right] \\ &= \bigoplus_i \left[(1 + a + x_i) - (1 + a) - (1 + x_i)\right] \\ &= \left[\bigoplus_i (1 + a + x_i)\right] - 3(1 + a) - \bigoplus_i (1 + x_i) \\ &= \left[\bigoplus_i (1 + a + x_i)\right] \oplus (1 + a) \oplus (1 + a^3)\end{aligned}$$

Expanding the first term and substituting for the symmetric functions in the x_i, we obtain

$$\begin{aligned}(1 + a) \otimes (1 + a^3) &= (1 + a + a^2) \oplus (1 + a) \oplus (1 + a^3) \\ &= 1 \text{ , the zero element.}\end{aligned}$$

The third product, by a similar calculation, is also zero.

////.

Summarising the above results, we have:

5.4.4 Proposition

The ring $\hat{G}(P_6(R))$ can be described by two generators x, y where $4x = 0 = 2y$, and with multiplication determined by the equations

$$x^2 = 2x, \; xy = 0, \; y^2 = 0.$$

////.

5.5 Proposition

If $n \leqslant 5$, let f be the integer part of $n/2$. Then $\tilde{K}(P_n(R))$ may be described by the generator x, and the two relations

$$x^2 = -2x \text{ and } x^{f+1} = 0$$

////.

The previous proposition is a very special case of a theorem of Adams ((5),Theorem 7.3) concerning K-theory of real projective spaces.

Let H be the complex line bundle over $P_n(R)$ which is the restriction of the Hopf bundle over $P_n(C)$ - complex projective n-space.

5.6 Proposition

(i) If E is a complex vector bundle over $P_2(R)$ (or $P_3(R)$) then E is stably equivalent to H or stably trivial according to whether $c_1(E)$ is non-zero or is zero.

(ii) If E is a complex vector bundle over $P_4(R)$ (or $P_5(R)$), then E is stably equivalent to

$$\underline{1} \; , H \; , H \oplus H \; , H \oplus H \oplus H$$

according to whether $c(E)$ is

$$1, \; 1 + a, \; 1 + a^2, \; 1 + a + a^2.$$

////.

5.7 By Adams result ((5), Theorem 7.3), $\tilde{K}(P_6(R))$ is cyclic, but $\hat{G}(P_6(R))$ by Prop.5.4.4 is not cyclic, which again shows that these two theories are not the same.(See 4.3)

6. The spectral sequence

In this section we interpret the results of section 4 in terms of the Atiyah-Hirzebruch spectral sequence as described in (8).

6.1 Let X be a finite CW complex, and let X^p be its p-skeleton. There is a filtration on $K^*(X)$ defined by

6.1.1 $$K^*_p(X) = \text{Kernel}\,[K^*(X) \longrightarrow K^*(X^{p-1})].$$

If X is connected, we have

6.1.2 $$K^*_1(X) \approx \tilde{K}^*(X)$$

This filtration has the property that

6.1.3 $$K^0_{2k-1}(X) = K^0_{2k}(X)$$

6.1.4 $$K^1_{2k+1}(X) = K^1_{2k}(X)$$

6.2 Proposition (Atiyah-Hirzebruch)

There is a spectral sequence $\{E^p(X)\}$ with $E^p_2(X) = H^p(X,Z)$ and $E^p_\infty(X) = K^*_p(X)/K^*_{p+1}(X)$ and with the following properties:

(i) the even differentials d_{2r} are zero

(ii) d_3 is the Steenrod operation Sq^3.

////.

6.3 We now consider the case in question when X is a CW complex with dim X $\leq$ 5. By 6.2 (i), $E^p_3 = E^p_2 = H^p(X,Z)$ and since $H^p(X,Z) = 0$ for $p \geq 6$ and

$$Sq^3 : H^p(X,Z) \longrightarrow H^{p+3}(X,Z)$$

is zero for $p<3$, it follows from 6.2 (ii) that $d_3 = 0$, so since $d_4 = 0$ we find that

$$E^p_5 = H^p(X,Z).$$

There is now only one possibly non zero differential, namely

$$d^0_5 : H^0(X,Z) \longrightarrow H^5(X,Z)$$

and we then find that

6.3.1 $$E^p_6 = E^p_\infty$$

Hence by Prop.6.2 we obtain the equations:

6.3.2
$$E_\infty^1 = H^1(X,Z) = K_1^*(X)/K_2^*(X) = K_1^1(X)/K_2^1(X)$$

$$E_\infty^2 = H^2(X,Z) = K_2^*(X)/K_3^*(X) = K_2^0(X)/K_3^0(X)$$

$$E_\infty^3 = H^3(X,Z) = K_3^*(X)/K_4^*(X) = K_3^1(X)/K_4^1(X)$$

$$E_\infty^4 = H^4(X,Z) = K_4^*(X)/K_5^*(X) = K_4^0(X)/K_5^0(X)$$

$$E_\infty^5 = H^5(X,Z)/\mathrm{Im}\ d_5^0 = K_5^*(X)/K_6^*(X) = K_5^1(X)$$

(using 6.1.3 and 6.1.4)

Again by using 6.1.3 and 6.1.4 with the above results, we obtain a short exact sequence

6.3.3
$$0 \longrightarrow H^4(X,Z) \longrightarrow \widetilde{K}(X) \longrightarrow H^2(X) \longrightarrow 0$$

[If dim $X \leqslant 4$, we see that $E_\infty^5 = 0$, and then we have a short exact sequence

6.3.4
$$0 \longrightarrow H^3(X,Z) \longrightarrow \widetilde{K}^1(X) \longrightarrow H^1(X,Z) \longrightarrow 0$$

Cor.4.8 says that 6.3.4 splits.]

Consider the sequence 6.3.3. Extensions of $H^4(X,Z)$ by $H^2(X,Z)$ are determined by cocycles

$$H^2(X,Z) \times H^2(X,Z) \longrightarrow H^4(X,Z).$$

Then Theorem 4.4 says that the extension which defines $\widetilde{K}(X)$ is represented as a cocycle as the cup product

$$H^2(X,Z) \times H^2(X,Z) \longrightarrow H^4(X,Z) .$$

7. Thom spaces

7.1 If $(X,A) \in \mathrm{Ob}\ \underline{C}^2$ then the inclusion $i:A \longrightarrow X$ is a proper map so induces a map of the one point compactifications, say $i^*:A^* \longrightarrow X^*$. Moreover, i^* is injective and A^* has the subspace topology, so we can consider $A^* \subseteq X^*$, and hence $(X^*,A^*) \in \mathrm{Ob}\ \underline{C}_c{}^2$.

Suppose E is a vector bundle over X (where E can be real or complex) Then since A is closed in X, and since the bundle is (by definition) locally trivial, we must have $E|A$ closed in E, so that $(E,E|A)$ is an object of $\underline{C}^2$.

7.2 Definition

The relative Thom space of E over (X,A), denoted by $(X,A)^E$ is

$$(X,A)^E = \frac{E^*}{(E|A)^*}$$

We note that

7.2.1 if $A = \emptyset$, then $(X,A)^E = E^*/(\emptyset)^* = E^*$, the usual Thom space

7.2.2 if $E = \underline{0}$, the zero bundle, then $(X,A)^{\underline{0}} = X^*/A^*$

7.2.3 if X is compact, $(X,A)^{\underline{0}} = X/A$ if $A \neq \emptyset$

$$(X,\emptyset)^{\underline{0}} = X^+.$$

7.3 Proposition

If (X,A) is compact and E is a vector bundle over X with associated disc bundle BE and associated sphere bundle SE, then

$$(X,A)^E \cong \frac{BE}{SE \cup B(E|A)}$$

In particular,

$$E^* = (X,\emptyset)^E \cong BE/SE.$$

Proof: Since X is compact, so are A, BE, SE. Let $f:[0,1) \longrightarrow [0,\infty)$ be a smooth, strictly increasing map such that $f(x)$ tends to infinity as x tends to 1, for example $x \mapsto \tan \frac{\pi x}{2}$. Then f defines a map $F:BE \longrightarrow E^*$ by the rule

$$F(e) = f(\|e\|).e \qquad \text{for } \|e\| < 1$$
$$\qquad = \text{base point of } E^* \qquad \text{for } \|e\| = 1.$$

Then F is well defined and continuous. Let

$$p : BE \dashrightarrow \frac{E^*}{(E|A)^*}$$

be the composite of F with the natural projection. Then p is injective except on $SE \cup BE|A$, which it maps to the base point. Hence p induces a continuous bijective map

$$\frac{BE}{SE \cup BE|A} \dashrightarrow \frac{E^*}{(E|A)^*}$$

which is a homomorphism since both spaces are compact.

////.

If (X,A) and $(Y,B) \in \mathrm{Ob}\ \underline{C}^2$, then their (categorical) product exists and is $(X \times Y,\ X \times B \cup A \times Y)$.

<u>7.4 Proposition</u>

<u>If</u> E <u>is a bundle over</u> (X,A) <u>and</u> F <u>is a bundle over</u> (Y,B) <u>then</u>

$$(\ (X,A) \times (Y,B)\ )^{E \times F} \cong (X,A)^E \wedge (Y,B)^F$$

<u>Proof</u>: Let $h : E^* \times F^* \longrightarrow (E \times F)^*$ be the natural map. Then h induces a map on quotient spaces:

$$\begin{tikzcd}
(E^* \times F^*) \arrow[r, "h"] \arrow[d] & (E \times F)^* \arrow[d] \\
\dfrac{E^*}{(E|A)^*} \times \dfrac{F^*}{(F|B)^*} \arrow[r, dashed, "k"] & \dfrac{(E \times F)^*}{(E \times F|X \times B \cup A \times Y)^*}
\end{tikzcd}$$

Then k is injective except on the wedge, which it maps to the base point. Thes k induces a continuous bijection

$$(X,A)^E \wedge (Y,B)^F \longrightarrow (\ (X,A) \times (Y,B)\ )^{E \times F}$$

which is a homeomorphism since both spaces are compact.

////.

We note that this homeomorphism is natural.

7.5 Let X, Y be differentiable manifolds with X a compact submanifold of Y. Let N be the normal bundle, and let E be any bundle over Y. Let $p:N \longrightarrow X$ be the bundle projection, which we think of as the tubular neighbourhood.

7.6 Proposition

$$(E|N)^* = (N \oplus (E|X))^*$$

Proof: The inclusion $N \subseteq Y$ (as the tubular neighbourhood) is homotopic to the composite

$$N \xrightarrow{p} X \xrightarrow{i} Y$$

so that the bundles $E|N$ and $p^*i^*E = p^*(E|X)$ over N are isomorphic. From the diagram

$$\begin{tikzcd} p^*(E|X) \arrow[r] \arrow[d] & E|X \arrow[d] \\ N \arrow[r, "p"] & X \end{tikzcd}$$

which is a pull-back, we see that as a topological space $p^*(E|X)$ is $N \oplus E|X$, from which the result follows.

////.

7.7 For such a situation, we define the Thom map

$$T_E : Y^E \longrightarrow X^{N \oplus i^*E}$$

as the map

$$Y^E = E^* \longrightarrow \frac{E^*}{E^* \setminus (E|N)} = (E|N)^* = X^{N \oplus i^*E}$$

The (based) homotopy class of this map is well defined, that is it does not depend on the choice of tubular neighbourhood. Moreover, if we choose a tubular neighbourhood, then the following diagram commutes:

7.7.3

$$\begin{tikzcd}
Y^{F \oplus F} \arrow[r, "\Delta"] \arrow[d, "T_E \oplus F"'] & Y^E \wedge Y^F \arrow[dr, "T_E \wedge 1"] & \\
& & X^{N \oplus i^*E} \wedge Y^F \\
X^{N \oplus i^*E \oplus i^*F} \arrow[r, "\Delta"] & X^{N \oplus i^*E} \wedge X^{i^*F} \arrow[ur, "1 \wedge i"'] &
\end{tikzcd}$$

where E,F are bundles on Y, and the map $i : X^{i^*F} \longrightarrow Y^F$ is induced by the inclusions $X \longrightarrow Y$ and $F|X \longrightarrow F$. The map

$$\Delta : Y^{E \oplus F} \longrightarrow Y^E \wedge Y^F$$

is the composite

$$Y^{E \oplus F} \longrightarrow (Y \times Y)^{E \times F} \xrightarrow{\cong} Y^E \wedge Y^F .$$

The fact that this diagram commutes will be needed later.

8. Orientation

8.1 Let $(\hat{t}, \hat{m})$ be a multiplicative homotopy functor. For $(X,A) \in \mathrm{Ob}\ \underline{C}^2$ define

8.1.1
$$t(X,A) = \hat{t}(X^*/A^*)$$
$$= \hat{t}((X,A)^{\underline{0}}) \quad \text{by } 7.2.2$$

If $f:(X,A) \longrightarrow (Y,B)$ is a proper map in $\underline{C}^2$, it induces a map $f^* : X^*/A^* \longrightarrow Y^*/B^*$ and hence a map $f^* : t(Y,B) \longrightarrow t(X,A)$.
It follows from Prop. 7.4 that $\hat{m}$ induces a natural associative and commutative multiplication

$$m(\ (X,A)\ ,\ (Y,B)\ ) : t(X,A) \otimes t(Y,B) \longrightarrow t(\ (X,A) \times (Y,B)\ )$$

as the (unique) map making the following diagram commute:

$$\begin{tikzcd}
t(X,A) \otimes t(Y,B) \arrow[rr, dashed] \arrow[d, "\cong"] & & t(\ (X,A) \times (Y,B)\ ) \arrow[d, "\cong"] \\
\hat{t}(X^*/A^*) \otimes \hat{t}(Y^*/B^*) \arrow[drr, "\hat{m}(X^*/A^*,Y^*/B^*)"'] & & t(X \times Y,\ X \times B \cup A \times Y) \arrow[d, "\cong"] \\
& & \hat{t}(X^*/A^* \wedge Y^*/B^*)
\end{tikzcd}$$

In particular, by taking $A = \emptyset$ and composing with the diagonal map we see that $t(X) = t(X,\emptyset) = \hat{t}(X^*)$ is a commutative ring, and by taking $(X,A) = (X,\emptyset)$ and $(Y,B) = (X,A)$ we see that $t(X,A)$ is a $t(X)$ module.

8.2 Proposition

If X is compact, $t(X)$ is a ring with identity, and $t(X,A)$ is a unital $t(X)$ module.

Proof: Since X is compact, $X^* = X^+$. Let $p: X^+ \longrightarrow S^0$ be the map taking base point to base point, and X to the other point.

Then $p^*: t(+) \longrightarrow t(X)$ is a ring homomorphism, and if 1 is the identity in $t(+)$, then $p^*(1)$ is an identity in $t(X)$, as can be seen from the commutative diagram:

8.2.1
$$\begin{tikzcd}
t(+) \otimes t(X) \arrow[r, "p^* \otimes 1"] \arrow[d, "\hat{m}(S^0, X^+)"'] & t(X) \otimes t(X) \arrow[d, "\hat{m}(X^+, X^+)"] \\
\hat{t}(S^0 \wedge X^+) \arrow[r, "(p \wedge 1)^*"] \arrow[dr] & \hat{t}(X^+ \wedge X^+) \arrow[d, "d^*"] \\
 & \hat{t}(X^+) = t(X)
\end{tikzcd}$$

(From axiom 1.3.7, the composite $\searrow$ which makes $t(X)$ into a $t(+)$ module is such that it is a unital $t(+)$ module.)

The proof that $t(X,A)$ is a unital $t(X)$ module is analogous.

////.

8.3 The multiplication defined above is a special case of a multiplication defined between Thom spaces. Suppose that E is a bundle over (X,A) and F is a bundle over (Y,B). Then $\hat{m}$ induces an associative commutative multiplication

$$\hat{t}((X,A)^E) \otimes \hat{t}((Y,B)^F) \longrightarrow \hat{t}(\,[(X,A) \times (Y,B)]^{E \times F}\,)$$

which is defined to be the composite

$$\begin{tikzcd}
\hat{t}((X,A)^E) \otimes t((Y,B)^F) \arrow[r, "\hat{m}"] \arrow[dr, dashed] & \hat{t}((X,A)^E \quad (Y,B)^F) \arrow[d, dashed, "\cong"] \\
& \hat{t}([(X,A) \times (Y,B)]^{E \times F})
\end{tikzcd}$$

(The multiplication defined in 8.1 can be obtained from this by putting E and F equal to the zero bundles.)

In particular, we see that $\hat{t}((X,A)^E)$ is a $t(X,A)$ algebra, or writing this another way, $t(E,E|A)$ is a $t(X,A)$ algebra. Also if E,F are bundles over X then this multiplication induces a multiplication

8.3.1 $$t(E) \otimes t(F) \longrightarrow t(E \times F)$$

which when followed by the diagonal map defines a multiplication

8.3.2 $$t(E) \otimes t(F) \longrightarrow t(E \oplus F)$$

8.4 Definition

Let E be a vector bundle over (X,A). Then E is orientable with respect to $\hat{t}$ over (X,A) if there is a $t(X,A)$ module isomorphism

$$t(X,A) \longrightarrow t(E,E|A)$$

Such an isomorphism is called an orientation of E over (X,A). In particular, if $A = \emptyset$, then a vector bundle over X is orientable if there is a $t(X)$ module isomorphism

$$t(X) \longrightarrow t(E)$$

8.5 Proposition

If X is compact, and E is a bundle over X orientable with respect to $\hat{t}$, then there is a bijective corresponeence (not natural) between the orientations of E and the units of $t(X)$.

Proof: Let $\eta : t(X) \longrightarrow t(E)$ be an orientation of E, and let u be a unit in $t(X)$. Then multiplication by u is a $t(X)$ module isomorphism of $t(X)$, and so the composite $\eta u : t(X) \longrightarrow t(E)$ is an orientation. This construction defines a function from the set of units to the set of orientations. Conversely, if $\zeta : t(X) \longrightarrow t(E)$ is an orientation, then $\eta^{-1}\xi : t(X) \longrightarrow t(Y)$ is a $t(X)$ module isomorphism and so is multiplication by some unit, say v. The function $\xi \longmapsto v$ maps orientations to units and is the inverse of the function defined above.

////.

9. The umkehr homomorphism

Let $i:X \longrightarrow Y$ be an embedding of (differentiable or complex analytic) manifolds in $\underline{C}$.

9.1 Definition

$i:X \longrightarrow Y$ is a $\hat{t}$ embedding if the normal bundle is $\hat{t}$ orientable. (Remark: under this definition it is not necessarily true that the composite of 2 $\hat{t}$ embeddings is a $\hat{t}$ embedding. This difficulty is remedied by the introduction of 'classes of vector' bundles in section 11.)

Suppose that $i:X \longrightarrow Y$ is a $\hat{t}$ embedding. Let

$$\eta : t(X) \longrightarrow t(N)$$

be an orientation, and let

$$T: Y^* \longrightarrow X^N = N^*$$

be the Thom map (see 7.7). We define $i(\eta):t(X) \longrightarrow t(Y)$ to be the composite

$$t(X) \xrightarrow[\eta]{} t(N) \xrightarrow[t(T)]{} t(Y)$$

9.3 Proposition

$i(\eta)$ is a $t(X)$ module homomorphism, i.e. if $a \in t(X)$ and $b \in t(Y)$ then

$$i(\eta)(a.i^*b) = (i(\eta)a).b$$

Proof: Since η is a $t(X)$ module homomorphism, it suffices to prove that $t(T)$ is a $t(X)$ module homomrphism. Consider the diagram:

9.3.1
$$\begin{tikzcd}
\hat{t}(X^*) \otimes \hat{t}(X^N) \arrow[r, dashed, "i^* \otimes T^*"] \arrow[d, dashed, "\hat{m}"'] & \hat{t}(Y^*) \otimes \hat{t}(Y^*) \arrow[d, "\hat{m}"] \\
\hat{t}(X^* \wedge X^N) \arrow[r, "(i \wedge T)^*"] \arrow[d, "\Delta^*"'] & \hat{t}(Y^* \wedge Y^*) \arrow[d, "\Delta^*"] \\
\hat{t}(X^N) \arrow[r, "T^*"] & \hat{t}(Y^*)
\end{tikzcd}$$

The top square commutes by naturality of multiplication, and the bottom square commutes by naturality of $\hat{t}$. The commutativity of the outside square says that T^* is a $t(X)$ module homomrphism.

////.

9.4 Proposition

If $i:X \longrightarrow Y$ *is a* $\hat{t}$ *embedding with* X *compact then*

$$i(\eta)(ab) \cdot i(\eta)(1) = i(\eta)(a) \cdot i(\eta)(b)$$

Proof: Since $t(T) = T^*$ is a ring homomorphism, it suffices to show that $\eta(ab).\eta(1) = \eta(a).\eta(b)$. This is a consequence of the fact that η is a $t(X)$ module map by the following Lemma

9.5 Lemma (Riemann-Roch)

Let R be a commutative ring with identity, and A be a unital R algebra. If $f:R \longrightarrow A$ is an isomorphism of R modules, define the Euler class $e(f) = f^{-1}((f(1))^2)$. Then

(i) $f(r) = rf(1)$

(1i) $f(rr')f(1) = f(r)f(r') = f(rr'e)$

Proof: (i) Trivial.

(ii) $f(r)f(r') = rf(1)r'f(1)$ by (i)

$= rr'f(1)f(1)$ by commutativity

$= f(rr')f(1)$ by (i)

$rr'f(1)f(1) = rr'f(e)$ by definition of e

$= f(rr'e)$

////.

10 A Riemann-Roch Theorem

10.1 Let $\hat{\alpha}:\hat{t} \longrightarrow \hat{s}$ be amap of MHF, and let $i:X \longrightarrow Y$ be an embedding of compact manifolds whose normal bundle, say N, is orientable with respect to both $\hat{t}$ and $\hat{s}$. Let

$$\eta : t(X) \longrightarrow t(N)$$
$$\zeta : s(X) \longrightarrow s(N)$$

be orientations. Since X is compact, t(X) has an identity (Prop 5.2) and we define $\tau \in s(X)$ by

$$\tau = \zeta^{-1}\alpha(N)\,\eta(1).$$

10.2 Proposition(Riemann-Roch)

$$\alpha(Y)i(\eta)(x) = i(\zeta)(\alpha(X)(x).\tau) \quad \text{for } x \in t(X)$$

Proof.

Consider the diagram:

$$\begin{tikzcd}
t(X) \arrow[r] \arrow[d, "\alpha(X)"'] & t(N) \arrow[r, "t(T)"] \arrow[d, "\alpha(N)"] & t(Y) \arrow[d, "\alpha(Y)"] \\
s(X) \arrow[r, "\zeta"'] & s(N) \arrow[r, "s(T)"'] & s(Y)
\end{tikzcd}$$

$$\begin{aligned}
i(\zeta)(\alpha(X)(x).\tau) &= s(T)\zeta(\alpha(X)(x).\tau) \\
&= s(T)(\alpha(X)(x).\alpha(N)\eta(1)) \\
&= s(T)\alpha(N)(x.\eta(1)) \\
&= \alpha(Y)t(T)\eta(x) \\
&= \alpha(Y)i(\eta)(x).
\end{aligned}$$

////.

10.3 For example, let $\hat{t} = \tilde{K}$ and $\hat{s} = \bigoplus_{n>0} \tilde{H}^{2n}(-,Q)$ and let $\hat{\alpha}$ be the Chern Character.

Suppose $i:X \longrightarrow Y$ is a differentiable embedding between compact differentiable manifolds whose normal bundle N (of dimension 2n say) is almost complex. If we choose a complex structure on N this determines an orientation

$$\lambda_N : K(X) \longrightarrow K(N)$$

namely the Bott (or Thom) isomorphism, (see Chapter 2, no.1.6). Since N is almost complex, it is orientable with respect to integer cohomology and the Thom class $U \in H^{2n}(X^N, \mathbb{Z})$ can be chosen (there is a choice of two) such that the image of U under the map

$$H^{2n}(X^N) \longrightarrow H^{2n}(BN) \xrightarrow{\cong} H^{2n}(X)$$ is the Euler class of N, i.e. the n-th Chern class of the complex structure.

In order to apply Prop.10.2, we have only to interpret the element τ, which is the Todd class of N, since

$$\mathfrak{Z}(\tau) = ch\,\lambda_N = \mathfrak{Z}(\text{Todd class of N}) \text{ and } \mathfrak{Z} \text{ is injective.}$$

10.4 Corollary

Let $i:X \longrightarrow Y$ *be a differentiable embedding between compact differentieble manifolds whose normal bundle* N *is almost complex. Then if* F *is a complex vector bundle over* X,

$$ch\; i(\lambda_N)(F) = i_*(ch(F).Todd(N))$$

where $i_*:H^*(X,Q) \longrightarrow H^*(Y,Q)$ *is the Gysin homomorphism, and* Todd(N) *is the Todd class of the complex structure on* N.

////.

10.5 For the purposes of the next chapter, we wish to obtain the corresponding Riemann-Roch Theorem for Chern classes, and so we consider the map $\mathcal{C}:\hat{K} \longrightarrow \hat{G}$ (see 3.20). Let N be a complex vector bundle over X, and as in 10 3, let $U \in H^{2n}(X^N, \mathbb{Z})$ be the Thom class. Define $\mathfrak{Z}:G(X) \longrightarrow \hat{G}(X^N)$ by

$$[n, 1 + a_1 + \quad \dots + a_r] \longrightarrow [0,\ 1 + nU + a_1U + \ \dots + a_rU]$$

In general, is not even additive, but it is bijective. Moreover if we identify G(X) with $\prod_{m\geq 0} H^{2m}(X,\mathbb{Z})$ as sets (see 3.8) and if $i:X \longrightarrow Y$ is a differentiable embedding between compact differentiable manifolds whose normal bundle is almost complex, then the composite

$$\begin{tikzcd} H^{ev}(X,\mathbb{Z}) \arrow[rr, dashed] \arrow[d, equal] & & H^{ev}(Y,\mathbb{Z}) \arrow[d, equal] \\ G(X) \arrow[r] & \hat{G}(X^N) \arrow[r, "\hat{G}(T)"] & G(Y) \end{tikzcd}$$

is precisely the Gysin homomorphism.

Suppose F is a complex vector bundle over X. Then $\hat{C}(F.\lambda_N)$ is an element of $\hat{G}(X^N)$ so can be written uniquely

$$\hat{C}(F.\lambda_N) = [0,\ 1 + a_0U + \ldots + a_rU]$$

where $a_r \in H^{2r}(X)$. Consider the commutative diagram:

$$\begin{array}{ccccccc} K(X) & \xrightarrow{\lambda} & \tilde{K}(X^N) & \longrightarrow & K(BN) & \longrightarrow & K(X) \\ & & \downarrow{\scriptstyle \hat{C}} & & \downarrow{\scriptstyle C} & & \downarrow{\scriptstyle C} \\ G(X) & \longrightarrow & \hat{G}(X^N) & \longrightarrow & G(BN) & \longrightarrow & G(X) \end{array}$$

Now $F \mapsto$ to $C(F.\lambda_{-1}N^\vee)$, where $\lambda_{-1}N^\vee = \sum_{r\geq 0}(-1)^r \lambda^r N^\vee$ is the image of $\lambda_N(1)$ in $K(X)$. But $F \mapsto$ to $[0, 1 + a_0 e + \ldots + a_r e]$ where $e = c_n(N)$ is the Euler class. Hence

$$C(F.\lambda_{-1}N^\vee)) = [0, 1 + a_0 e + \ldots + a_r e]$$

and so

$$\hat{C}(F.\lambda_N) = \frac{C(F.\lambda_{-1}N^\vee)}{c_n(N)}$$

where the expression on the right hand side denotes the element

$$[a_0,\ 1 + a_1 + \ldots. + a_r] \in G(X)$$

<u>10.6 Corollary</u>

<u>Under the axioms of Cor. 10.4,</u>

$$C(i(\lambda_N)(F)) = i_*\left(\frac{C(F.\lambda_{-1}N^\vee)}{c_n(N)}\right)$$

////.

<u>10.7</u> We see that there can be no formula analogous to that of Cor. 10.6 involving a 'Todd class' as multiplier. For suppose that there is an element $\tau \in G(X)$ such that

$$C(i(\lambda_N)(F)) = i_*(C(F).\tau)$$

Putting $F = 1$ implies that $\tau = C(\lambda_{-1}N^\vee)/c_n(N)$ so that we could write

$$C(i(\lambda_N)(F) = i_*(C(F).\frac{C(\lambda_{-1}(N^\vee))}{c_n(N)} \quad \text{(see (9))}$$

By considering a specific example, we see that such a formula is false.

Let $X = P_2(\mathbb{C})$ and let N be the Hopf bundle. Let $Y = BN$, an let $i\ X \longrightarrow Y$ be the inclusion of the zero section. This has normal bundle $N = H$. Let $F = H$ also. If the equation 10.7.1 were correct, then

$$i^*C(i_*H) = i^*i_*\left(C(H).\frac{C(\lambda_{-1}H^\vee)}{c_1(H)}\right)$$

$$= C(i^*i\ (H) = C(H.\ \lambda_{-1}H^\vee)$$

$$= C(H - 1) = [0, 1 + x]$$

where x generates $H^2(P_2(C))$. However

$$\frac{C(\lambda_{-1}H^\vee)}{c_1(H)} = \frac{1}{x}\left[0,\ 1 + x + x^2 + x^3 + \ldots\right]$$

$$= [1,\ 1 + x + x^2] \qquad \text{since } x^3 = 0.$$

From this a simple calculation shows that

$$i^*i_*\left(C(H).\frac{C(\lambda_{-1}H^\vee)}{x}\right) = i^*i_*[1, 1 + 2x + x^2]$$

$$= [0,\ 1 + x + 2x^2]$$

and $2x^2 \neq 0$.

10.8 An important case of Cor.10.6 is when $X \hookrightarrow Y$ has complex codimension 1. Let $C(N) = [1, 1 + v]$, and let $C(F) = [n, 1 + c_1 + \ldots + c_n]$. Then $C(\lambda_{-1}N^\vee) = -[0, 1-v]$ and

$$C(F.\ \lambda_{-1}N^\vee) = C(F).C(\lambda_{-1}N^\vee)$$

If we formally factorise $C(F) = \sum [1, 1 + b_i]$, then

$$C(F.\ \lambda_{-1}N^\vee) = \sum [0, 1 + b_i] - \sum [0, 1 + b_i - v]$$

$$= \frac{C(F)}{\sum (1 - v)^i\ c_{n-i}}$$

and so we obtain the equation

10.8.1 $$C(i_*F) = 1 + i_*\left(\frac{1}{v}\right)\left[\frac{C(F)}{\sum(1-v)^i c_{n-i}} - 1\right].$$

11 Classe of vector bundles

11.1 A collection of vector bundles (real or complex) $\underline{V}$ over spaces in $\underline{C}$ is admissible if the following four conditions are satisfied:

(i) $\underline{V} \subset \underline{C}$ i.e. if $p:E \longrightarrow X$ is a vector bundle in $\underline{V}$ then p is a map in $\underline{C}$.

(ii) if $E \longrightarrow X$ is in $\underline{V}$ and $f:Y \longrightarrow X$ is a map in $\underline{C}$, then

$f^*E \longrightarrow Y$ is in $\underline{V}$.

(iii) if $E \longrightarrow X$ and $F \longrightarrow Y$ in $\underline{V}$ then so is $E \times F \longrightarrow X \times Y$.

(iv) if $X \in \underline{C}$, the zero bundle $\underline{0} \longrightarrow X$ is in $\underline{V}$.

we note that if E and F are vector bundles over X and belong to $\underline{V}$, then $E \oplus F$ is in $\underline{V}$, and so is $E \backslash A$ for any $A \subseteq X$.

11.2 Examples

11.2.1 $\underline{V}$ = real bundles of dimension a multiple of n

11.2.2 $\underline{V}$ = complex vector bundles

11.2.3 $\underline{V}$ = collection of all complex bundles E such that $c_i(E) = 0$ for $1 \leq i \leq n$

11.2.4 $\underline{V}$ = real bundles which admit a spin-reduction.

11.3 Definition Let $\mathcal{E}$ be a MHF. A natural $\mathcal{E}$ orientation for an admissible collection $\underline{V}$ comprises, for each $E \longrightarrow X$ in $\underline{V}$ a $t(X)$ module homomorphism

$$\eta_E : t(X) \longrightarrow t(E) \qquad \text{such that}$$

(i) if X is compact, η_E is an isomorphism (i.e. an orientation)

(ii) if $E \longrightarrow X$ is in $\underline{V}$ and $f:Y \longrightarrow X$ is a proper map in $\underline{C}$ then the diagram

$$\begin{tikzcd} t(X) \arrow[r, "\eta_E"'] \arrow[d, "f^*"'] & t(E) \arrow[d, "f^*"] \\ t(Y) \arrow[r, "\eta_{f^*E}"'] & t(f^*E) \end{tikzcd}$$

commutes.

(iii) if $\underline{0} \longrightarrow X$ is the zero bundle, then $\eta_{\underline{0}}:t(X) \longrightarrow t(X)$ is the identity.

(iv) if $E \longrightarrow X$ and $F \longrightarrow Y$ in $\underline{V}$ then the diagram

$$\begin{tikzcd} t(X) \otimes t(Y) \arrow[r, "\eta_E \otimes \eta_F"] \arrow[d, "m"] & t(E) \otimes t(F) \arrow[d, "m"] \\ t(X \times Y) \arrow[r, "\eta_{E \times F}"'] & t(E \times F) \end{tikzcd}$$ commutes.

(The vertical maps are the multiplications defined in 8.1 and 8.3)

11.4 Examples

11.4.1 Given a MHF $\hat{t}$, there may be no class (other than the class of zero bundles) which is naturally $\hat{t}$ orientable.

11.4.2 Given a class of vector bundles there may be no MHF other than the zero theory with respect to which the class is naturally orientable.

11.4.3 For the class of complex vector bundles, there is a natural orientation with respect to K-theory and with respect to $H^{ev}(-)$.

11.4.4 For the class of real bundles admitting a spin reduction there is a natural orientation wôth respect to KO-theory.

11.5 Let $i:X \longrightarrow Y$ be an embedding betwe n compact differentiable manifolds in $\underline{C}$. We say that i is a $\underline{V}$ - embedding if the normal bundle is in $\underline{V}$. Let $\hat{t}$ be a MHF, and let η be a natural $\hat{t}$ orientation for $\underline{V}$. Define $i_*:t(X) \longrightarrow t(Y)$ to be $i(\eta)$ (see 9.1).

11.6 Proposition

Let $i:X \longrightarrow Y$ *and* $j:Y \longrightarrow Z$ *be* $\underline{V}$ *embeddings. Then so is* ji *and*

$$(ji)_* = j_* i_*$$

Proof:Let N be the normal bundle of i and M of j. Then $N \oplus i^*M$ is the normal bundle of ji, and so is in $\underline{V}$. Let $T_j:Z^* \longrightarrow Y^M$ and $T_M:Y^M \longrightarrow X^{N \oplus i^*M}$ be Thom maps so that their composite is a Thom map $T_{ji}:Z^* \longrightarrow X^{N \oplus i^*M}$. (By 7.7 Thom maps are only defined upto homotopy.) By &.7.2, the following diagram commutes upto homotopy:

$$\begin{tikzcd} Y^M \arrow[r, "d"] \arrow[d, "T_M"'] & Y^* \wedge Y^M \arrow[dr, "T_i \wedge 1"] & \\ & & X^N \wedge Y^M \\ X^{N \oplus i^*M} \arrow[r, "d"'] & X^N \wedge X^{i^*M} \arrow[ur, "1 \wedge T_i"'] & \end{tikzcd}$$

Now consider the diagram

$$\begin{tikzcd}
\mathcal{E}(X^*) \arrow[d, "\eta_N"'] \arrow[drr, dashed, "\eta_{N \oplus i^*M}"] & & & \\
\mathcal{E}(X^N) \arrow[rr, "\cdot\eta_{i^*M}(1)"] \arrow[dr, "\cdot\eta_M(1)"'] \arrow[dd, "T_i^*"'] & & \mathcal{E}(X^N \oplus i^*M) \arrow[dd, "T_M^*"] \arrow[ddr, "T_{ji}^*"] & \\
 & \mathcal{E}(X^N \wedge Y^M) \arrow[ur, "d^*(1\wedge T_i)^*"] \arrow[dr, "d^*(T_i\wedge 1)^*"'] & & \\
\mathcal{E}(Y^*) \arrow[rr, "\eta_M"'] & & \mathcal{E}(Y^M) \arrow[r, "T_j^*"'] & \mathcal{E}(Z^*)
\end{tikzcd}$$

where $\cdot\, \eta_M(1)$ is the map induced by the multiplication

$$\mathcal{E}(X^N) \otimes \mathcal{E}(Y^M) \longrightarrow \mathcal{E}(X^N \wedge Y^M)$$

on the right by the element $\eta_M(1) \in \mathcal{E}(Y^M)$. The above diagram is commutative fcr the following reasons:

Triangle (a) by 11.3 (ii) and (iv)

Triangle (b) by 11.3 (ii) and (iv)

Triangles (c) and (d) by naturality of $\mathcal{E}$

Square (e) by 11.3 (ii), (iii) and (iv).

The composite map $\downarrow\!\rightarrow$ is $j_* i_*$ and the map $\searrow$ is $(ji)_*$ which completes the pro f of the proposition.

////.

11.7 Definition A $\underline{V}$-homotopy between two $\underline{V}$-embeddings i_o, i_1 of X in Y is a level preserving V-embedding $H : X \times I \longrightarrow Y \times I$ with $H_0 = i_0$ and $H_1 = i_1$.

<u>11.8 Proposition</u>

If $i_0, i_1 : X \longrightarrow Y$ are V-homotopic V-embeddings then $i_{0*} = i_{1*}$.

<u>11.7 Definition</u> A <u>V</u>-homotopy between two <u>V</u>-embeddings i_0, i_1 of X in Y is a level preserving <u>V</u>-embedding $H: X \times I \longrightarrow Y \times I$ such that there exist $\varepsilon_0, \varepsilon_1 > 0$ such that

$$H(x,t) = (i_0(x), t) \quad \text{for } 0 \leqslant t \leqslant \varepsilon_0$$

and $$H(x,t) = (i_1(x), t) \quad \text{for } 1 - \varepsilon_1 \leqslant t \leqslant 1$$

<u>11.8 Remark</u> We note that <u>V</u>-homotopy is an equivalence relation which implies ordinary homotopy. Under certain circumstances, for example if <u>V</u> is all vector bundles and we take smooth embeddings of manifolds, then <u>V</u>-homotopy will be equivalent to the usual notion of homotopy for embeddings.

<u>11.9 Proposition</u>

<u>If</u> i_0, i_1 <u>are V-homotopic V-embeddings of</u> X <u>in</u> Y, <u>then</u>

$$i_{0*} = i_{1*} .$$

<u>Proof:</u> Let H be a V-homotopy between i_0 and i_1. Let N be the normal bundle of $H: X \times I \longrightarrow Y \times I$. Then since near the <u>level</u> zero H is i_0, the normal bundle of $i_0: X \longrightarrow Y$ is $j_0{}^*N$, where $j_n: X \longrightarrow X \times I$ for $n = 0,1$ are the inclusions of X on the ends. Consider the diagram

$$\begin{tikzcd}
\mathcal{E}(X^*) \arrow[r, "\eta_{j_0{}^*N}"'] & \mathcal{E}(X^{j_0{}^*N}) \arrow[r, "T_{i_0}"'] & \mathcal{E}(Y^*) \\
\mathcal{E}((X \times I)^*) \arrow[u, "j_0^*"] \arrow[r, "\eta_N"'] & \mathcal{E}((X \times I)^N) \arrow[u, "j_0^*"'] \arrow[r, "T_H"'] & \mathcal{E}((Y \times I)^*) \arrow[u, "j_0^*"']
\end{tikzcd}$$

which commutes by naturality. Thus the map i_{0*} is the composite

$$\mathcal{E}(X^*) \xrightarrow[p^*]{} \mathcal{E}((X \times I)^*) \xrightarrow[\eta_N]{} \mathcal{E}((X \times I)^N) \xrightarrow[T_H]{} \mathcal{E}((Y \times I)^*) \xrightarrow[p^{*-1}]{} \mathcal{E}(Y^*)$$

where p denotes a projection map. But then this map similarly equals i_{1*}.

////.

Appendix to Chapter 1: The real vers ion of section 4

A.1 The first attempt at deriving the real version of section 4 is to define $\hat{GO}(X)$ to be the set of formal power series

$$1 + x_1 + x_2 + \dots \qquad \text{where } x_i \in H^i(X,\mathbb{Z}_2)$$

and with addition and multiplication defined in analogous fashion to that for $\hat{G}$. Then the Stiefel-Whitney classes define a map

$$w:KO \longrightarrow \hat{GO}.$$

However, since $KO(S^{4n}) = \mathbb{Z}$ while $GO(S^{4n}) = \mathbb{Z}_2$ we see that this homomorphism cannot carry much information. In order to remedy this (especially in dimensions of the form 4n) we try to introduce Pontrjagin classes, but since these do not obey a'Whitney sum' formula, we cannot define them on KO. However, we can obtain a real version in low dimensions by considering both Stiefel-Whitney and Pontrjagin classes, which we describe briefly. We work over the category $\underline{W}_*^f$ as in 4.15.

A.2 Let $\hat{GP}(X)$ be the set $\hat{GO}(X) \times H^4(X,\mathbb{Z})$. We define addition by

$$((1 + x_1 + \dots),p) \oplus ((1 + y_1 + \dots),q)$$
$$= ((1 + z_1 + \dots),t)$$

where

$$z_n = \sum_{i+j=n} x_i y_j$$

$$t = p + (\beta x_1)(\beta y_1) + q$$

where $\beta:H^1(X,\mathbb{Z}_2) \longrightarrow H^2(X,\mathbb{Z})$ is the Bockstein homomorphism.

A.3 Proposition

(i) $\hat{GP}(X)$ is an abelian group.

(ii) $\hat{GP}$ is a half exact functor.

Proof (ii) is similar to Prop.3.10.

(i) Addition is associative follows since β is a homomorphism. The zero is the elemtnt (1,0). To find the inverse of $((1 + a_1 + \cdots),p)$ we first find elements $x_i \in H^i(X,\mathbb{Z}_2)$ such that

$(1 + a_1 + \dots) \oplus (1 + x_1 + \dots) = 1$ in $\hat{GO}(X)$.

In particular, $x_1 = a_1$. Let $q = p - (\beta(a_1))^2$. Then

$$((1 + a_1 + \dots),p) \oplus ((1 + x_1 + \dots),q) = (1,0).$$

////

A.4 Suppose $E \longrightarrow X$ is a real vector bundle, and let E^c be its complexification.

A.4.1 Proposition

$$c_1(E^c) = \beta w_1(E)$$

Proof: We can identify real line bundles with their first Stiefel-Whitney class in $H^1(X;\mathbb{Z}_2)$ and complex line bundles with their first Chern class in $H^2(X;\mathbb{Z})$. Moreover, with this identification, complexification of a real line bundle is the Bockstein

$$\beta : H^1(X;\mathbb{Z}_2) \longrightarrow H^2(X;\mathbb{Z}).$$

Then

$$c_1(E^c) = \det(E^c) = (\det E)^c = \beta w_1(E).$$

////

Define $q(E) \in \hat{GP}(X)$ to be the element

$$((1 + w_1(E) + w_2(E) + \dots),p_1(E))$$

where $w_i(E)$ is the i-th Stiefel-Whitney class, and $p_1(E)$ os the first Pontrjagin class.

A.4.2 Lemma $\quad q(E \oplus F) = q(E) + q(F)$

Proof: It suffices to check the component in $H^4(X,\mathbb{Z})$. Then

$$p_1(E \oplus F) = c_2((E \oplus F)^c) = p_1(E) + c_1(E^c)c_1(F^c) + p_1(F).$$

The result now follows from Prop.A.4.1.

////

Hence q induces a homomorphism $q:KO \longrightarrow \hat{GP}$, and we obtain a real version of Theorem 4.4 .

A.5 Theorem Let X be a finite CW complex of dimension ≤ 5. Then

$$q:\widetilde{KO}(X) \longrightarrow \hat{GP}(X)$$

is an isomorphism modulo 2-groups.

Proof: For n = 1,2,3,5, both $\widetilde{KO}(S^n)$ and $\widetilde{GP}(S^n)$ are 2-groups. For S^4 the map q is essentially the map

$$\widetilde{KO}(S^4) \longrightarrow H^4(S^4,\mathbb{Z}) \times H^4(S^4,\mathbb{Z}_2)$$

which takes a vector bundle E to $(p_1(E), w_4(E))$. It is well known (3) that $p_1(E)$ is a multiple of 2 (thinking of $H^4(S^4,\mathbb{Z})$ as $\mathbb{Z}$) and that every such multiple arises. Thus the cokernel is a 2-group, and the map is injective, so is an isomorphism module 2-groups.

////.

A.6 In order to remove the dimension restrictions and hence use the higher Pontrjagin classes, we need to know of the existence of higher analogues of the Bockstein map which would have to have similar properties to the Steenrod squares, and also provide a generalisation of Prop.A.4.1. The existence of such cohomology operations depends on a detailed study of the integer cohomology ring of an Eilenberg-Maclane space $K(\mathbb{Z}_2, 2n+1)$.

CHAPTER 2 : BLOWING UP SUBMANIFOLDS

Chapter 2. Blowing up Chern classes

In this chapter we apply the Riemann Roch Theorem of the previous chapter to prove a result of Porteous (10) concerning the effect on Chern classes and Chern character of blowing up a submanifold of a complex manifold. In section 1 we reproduce from Atiyah (1) a presentation of K-theory in a form suitable for our calculations. In section 2 we describe the process of blowing up in terms of a universal property, and in section 3 we prove the theorem.

1. K-theory

If $n \geqslant 1$, we define $C_n(X,A)$ to be the category as follows: an object of $C_n(X,A)$ is a collection $E_n,\ldots,E_0$ of bundles over X together with maps $\alpha_i : E_i|A \longrightarrow E_{i-1}|A$ such that

$$0 \longrightarrow E_n|A \xrightarrow{\alpha_n} \quad \cdots \qquad \cdots \longrightarrow E_1|A \xrightarrow{\alpha_1} E_0\, A \longrightarrow 0$$

is exact. The morphisms $\varphi : E \longrightarrow F$, where $E = (E_i, \alpha_i)$ and $F = (F_i, \beta_i)$ are collections of maps $\varphi_i : E_i \longrightarrow F_i$ such that

$$\beta_i \varphi_i = \varphi_{i-1} \alpha_i$$

In particular, $C_1(X,A)$ consists of pairs of bundles E_1, E_0 over X and isomorphisms $\alpha : E_1|A$ —— $E_0|A$

An elementary sequence in $C_n(X,A)$ is a sequence of the form $0,0,\ldots,0,E_p,E_{p-1},0,\ldots,0$ where $E_p = E_{p-1}$ and α = identity. We define $E \sim F$ if for some elementary objects $Q_1,\ldots,Q_n,P_1,\ldots,P_m$

$$E \oplus Q_1 \oplus \ldots \oplus Q_n \cong F \oplus P_1 \oplus \ldots \oplus P_m$$

The set of such equivalence classes is denoted by $L_n(X,A)$. It is clear that $L_n(X,A)$ is a semigroup for each n. There is a natural inclusion $C_n(X,A) \hookrightarrow C_{n+1}(X,A)$ which induces a homomorphism

$$L_n(X,A) \longrightarrow L_{n+1}(X,A).$$

1.2 Theorem

For $n \geq 1$, the maps $L_n(X,A) \longrightarrow L_{n+1}(X,A)$ are isomorphisms, and

$L_n(X,A) \cong K(X,A)$.

Proof: See Atiyah (11) Theorem 2.6.1. ////.

1.3 If E_1 and E_0 are bundles over X and $\alpha : E_1|A \longrightarrow E_0|A$ is an isomorphism, then the element of $K(X,A)$ determined by this data is denoted by $d(E_1,E_0;\alpha)$. Theorem 1.2 then says that this is a typical element.

1.4 Proposition.(Atiyah- Hirzebruch)

(i) $d(E_1,E_0;\alpha)$ is natural, i.e. if $f:(Y,B) \longrightarrow (X,A)$ then

$$d(f^*E_1,f^*E_0;f^*\alpha) = f^*d(E_1,E_0;\alpha)$$

(ii) $d(E_1,E_0;\alpha)$ depends only on the homotopy class of α.

(iii) if $A = \emptyset$, then $d(E_1,E_0;\alpha) = E_0 - E_1$

(iv) if $j:(X,\emptyset) \longrightarrow (X,A)$ is inclusion, then $j^*d(E_1,E_0;\alpha) = E_1 - E_0$

(v) $d(E_1,E_0;\alpha) = 0$ iff there is a vector bundle G over X such that $\alpha \oplus 1 : E_1 \oplus G|A \longrightarrow E_0 \oplus G|A$ extends to an isomorphism over X.

(vi) $d(E_1 \oplus F_1, E_0 \oplus F_0; \alpha \oplus \beta) = d(E_1,E_0;\alpha) + d(F_1,F_0;\beta)$

(vii) $d(E_1,E_0;\alpha) = -d(E_0,E_1;\alpha^{-1})$

(viii) if $\beta : E_0|A \longrightarrow F|A$ is an isomorphism, then

$$d(E_1,E_0;\alpha) + d(E_0,F;\beta) = d(E_1,F;\beta\alpha)$$

(ix) if D is a vector bundle on X then

$$d(E_1 \times D, E_0 \times D; \alpha \times 1) = d(E_1,E_0;\alpha).D$$

Proof: See Atiyah-Hirzebruch (12) Proposition 3.3. ////.

1.5 Now let $\underline{C}$ be an admissible category as in Chapter 1. Then $K:\underline{C}_c \longrightarrow$ Abelian groups is half exact and induces a multiplicative homotopy functor $\tilde{K}:C_{c*} \longrightarrow$ Abelian groups with multiplication induced by 'external tensor product'. By the methods of Chapter 1 this extends to a functor on $\underline{C}^2$ which (in this case) extends the definition on $\underline{C}_c^2$.

1.6 Suppose E is a complex vector bundle over $X \in \mathrm{Ob}\, \underline{C}_c$. Let $E^\vee$ denote the dual bundle, and let BE and SE denote as usual the disc and sphere bundles of E. Let $p: BE \longrightarrow X$ be the projection map. If E has complex fibre dimension n, then we can define in a natural way the 'exterior complex' of E, denoted by $\Lambda(E)$ which is an element of $C_n(BE,SE)$. It is the object whose j-th component is $\lambda^j p^* E^\vee = p^* \lambda^j(E^\vee)$, and with the map from the j-th component to the j-1 th component over the point $b \in SE$ is given by evaluation at b. This sequence thus defines an element of $K(BE,SE) = K(E)$ denoted by λ_E, and multiplication by λ_E defines an orientation $K(X) \longrightarrow K(E)$ called the Thom isomorphism, or Bott periodicity.

1.7 We now consider the specaal case which we shall need later on. Suppose that X is a cpmplex manifold, and $Y \subset X$ is a complex submanifold of codimension 1. Let L be the normal bundle, which has complex fibre dimension 1. Let

$$q:(BL,SL) \longrightarrow (\bar{U}, \partial U)$$

be a tuhular neighboudhood of Y in X, and let

$$k:(\bar{U}, \partial U) \longrightarrow (X, X \setminus U)$$

be the inclusion map. The exterior complex of L is

$$p^* L^\vee \xrightarrow[e]{} \underline{1}$$

where e maps the functional f over the point $b \in SL$ to the complex nurber $f(b)$ over $b \in SL$. This sequence determines the element

$$\lambda_L = d(p^* L^\vee, \underline{1}; e) \in K(L)$$

and from Prop.1.4 (ix) the Thom isomorphism is defined by

$$E \longmapsto E.\lambda_L$$

where $E.\lambda_L$ is represented by the sequence

$$p^* L^\vee \otimes p^* E \xrightarrow[e \otimes 1]{} \underline{1} \otimes p^* E$$

If $j: Y \longrightarrow X$ denotes inclusion, then the umkehr map $j_*: K(Y) \longrightarrow K(X)$ is defined to be the composite

$$K(Y) \xrightarrow[\lambda_L]{} K(L) = K(BL,SL) \xrightarrow[q^*]{} K(\bar{U}, \partial U) \xrightarrow[k^{*-1}]{} K(X, X \setminus U) \longrightarrow K(X)$$

2. Blowing up submanifolds.

2.1 Remark For the rest of this chapter, manifold will mean complex analytic manifold, and differentiable will mean complex differentiable. However the analogous results hold in real analytic case.

2.2 Let $f:(X,Y) \longrightarrow (A,B)$ be a differentiable map between pairs of manifolds. Then f induces by differentiation a map between the exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & TY & \longrightarrow & TX|Y & \longrightarrow & N_X(Y) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Tf} & & \downarrow{\scriptstyle Tf} & & \downarrow{\scriptstyle N(f)} & & \\ 0 & \longrightarrow & TB & \longrightarrow & TA|B & \longrightarrow & N_A(B) & \longrightarrow & 0 \end{array}$$

where $N_X(Y)$ denotes the normal bundle of Y in X, and N(f) denotes the map induced between normal bundles.

2.3 Let $\underline{M}$ be the category whose objects are pairs (X,Y) of manifolds Y closed in X and whose morphisms from (X,Y) to (A,B) are those differentiable maps $f:(X,Y) \longrightarrow (A,B)$ which satisfy the two following conditions:

(i) $Y = f^{-1}B$

(ii) $N(f):N_X(Y) \longrightarrow N_A(B)$ is a monomorphism.

If $f:(X,Y) \longrightarrow (S,B)$ and $g:(R,S) \longrightarrow (X,Y)$ satisfy these two conditions, then so does $fg:(R,S) \longrightarrow (A,B)$, and we define composition in this category to be the usual composition of functions,

For $(X,Y) \in \mathrm{Ob}\,\underline{M}$, define $\mathrm{codim}(X,Y) = \dim X - \dim Y$. We note from condition (ii) above that there are no maps from (X,Y) to (A,B) if $\mathrm{codim}(X,Y) > \mathrm{codim}(A,B)$. Let $\underline{M}_1 \subset \underline{M}$ be the full subcategory of objects of codimension 1.

2.4 Proposition

<u>$\underline{M}_1$ is a reflective subcategory of $\underline{M}$. The reflection functor $\underline{M} \rightsquigarrow \underline{M}_1$ is given by blowing up the submanifold.</u>

Proof: Let $(X,Y) \in \mathrm{Ob}\,\underline{M}$, and let $p:(X',Y') \longrightarrow (X,Y)$ be the result of blowing up Y. Then p is a map in $\underline{M}$, and $\mathrm{codim}(X',Y') = 1$.

Suppose that $f:(A,B) \longrightarrow (X,Y)$ is a map in $\underline{M}$, where $\operatorname{codim}(A,B) = 1$. Then we must show that there is a unique map $g:(A,B) \dashrightarrow (X',Y')$ lifting f:

$$\begin{tikzcd} & (X',Y') \arrow[d, "p"] \\ (A,B) \arrow[ur, dashed, "g"] \arrow[r, "f"'] & (X,Y) \end{tikzcd}$$

Let L denote the normal bundle of B in A, and N the normal bundle of Y in X. We define $g:A \longrightarrow X'$ by its values on $A \setminus B$ and B separately.

Suppose that $a \in A \setminus B$. Then $f(a) \in X \setminus Y$ since $B = f^{-1}Y$. Now $p:X' \setminus Y' \longrightarrow X \setminus Y$ is a diffeomorphism, so we have no choice but to put $g(a) = p^{-1}f(a)$.

Suppose $b \in B$. The fibre L_b of L over b is mapped monomorphically by $N(f)$ into $N_{f(b)}$ so its image, which is a one dimensional subspace, determines a point in the corresponding projective bundle of N over Y; that is a point in Y'. Define $g(b)$ to be this point. Then g is a map in $\underline{M}$, and is the unique map such that $pg = f$.

////.

2.5 Proposition

If X is compact, the map $p:X' \longrightarrow X$ is an identification, that is X is obtained from X' by identifying along the fibres.

Proof: Since X is compact so is Y, and hence so are X' and Y'. Let X'' be the space obtained from X' by identifying along the fibres, and let $h:X' \longrightarrow X''$ be the natural map. Since $p:X' \longrightarrow X$ respects the identification h, there is a unique map $r:X'' \longrightarrow X$ such that $p = rh$. Since p is surjective, so is r. Since r is injective, so $r:X'' \longrightarrow X$ is a bijective map between compact hausdorff spaces, so is a homeomorphism. ////.

2.6 Corollary

The diagram

$$\begin{tikzcd} Y' \arrow[r, "j"] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ Y \arrow[r, "i"'] & X \end{tikzcd}$$

is a pushout (in the toplogical category). ////.

2.7 Corollary

$X/Y \cong X'/Y'$

Proof: Th inclusion $Y \hookrightarrow X$ is isomorphic to the inclusion $Y \hookrightarrow X' \cup_g Y$, so the quotient space X/Y is homeomorphic with the q otient space $X' \cup_g Y/Y = X'/Y'$.

////.

2.8 We now look at the question of which objects in $\underline{M}_1$ are obtained by blowing up something of codimension greater that 1. There are 3 obvious necessary conditions for (X',Y'):

2.8.1 that Y' is the total space of a fibre bundle

$$P^n(C) \longrightarrow Y' \longrightarrow Y$$

(with $n \geqslant 1$) where Y is some manifold, and such that identifying along the fibres gives a manifold X.

2.8.2 This projective bundle is associated to a vector bundle. Using the notation of (4), this projective bundle is an element say ξ of the set $H^1(Y,PGL_n(C))$. The exact sequence

$$0 \longrightarrow C^* \longrightarrow GL_n(C) \longrightarrow PGL_n(C) \longrightarrow 0$$

defines an exact sequence of pointed sets

$$0 \longrightarrow H^1(Y,C^*) \longrightarrow H^1(Y,GL_n(C)) \longrightarrow H^1(Y,PGL_n(C)) \xrightarrow{\delta} H^2(Y,C^*)$$

so that ξ is associated to a projective bundle iff $\delta\xi = 0$.

From the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow C \xrightarrow{\exp} C^* \longrightarrow 0$$

and the fact that $H^i(Y,C) = 0$ for $i \geqslant 1$, we see that

$H^i(Y,C^*) \cong H^{i+1}(Y,\mathbb{Z})$, so that the sequence of sets becomes

$$0 \longrightarrow H^2(Y,\mathbb{Z}) \longrightarrow H^1(Y,GL_n(C)) \longrightarrow H^1(Y,PGL_n(C)) \xrightarrow{\delta} H^3(Y,\mathbb{Z})$$

i.e. ξ is associated to a vector bundle iff the obstruction

$$\delta\xi = 0 \in H^3(Y,\mathbb{Z})$$

(Since $H^2(Y,\mathbb{Z})$ is the group of line bundles, this exact sequence also tells us that two n-dimensional vector bundles E,F have the same associated projective bundle iff there is a line bundle L such that $E \cong F \otimes L$.)

2.8.3 Suppose the vector bundle N which has ξ as associated projective bundle has Chern classes $c_1,\ldots,c_n$. Let L be the normal bundle of Y' in X'. Then L is the Hopf bundle over $Y' = P(N)$. Let $a \in H^2(Y',\mathbb{Z})$ be the first Chern class of L. Then $H^*(Y')$ is an $H^*(Y)$ module with generators $1,a,a^2,\ldots,a^{n-1}$ subject to the single relation

$$\sum_i (g^*c_{n-i})a^i = 0 \qquad \text{where } g:Y' \longrightarrow Y$$

which imposes some restriction on $H^*(Y')$.

2.8.4 Example When can $(P^n(C),X')$ occur from blowing up somethimg of codimension $m > 1$? If a denotes a generator of $H^2(P^n(C),\mathbb{Z})$ and since $\dim N = m$, 2.8.3 says that there is a single relation of the form $x.a^m = 0$ for some integer x, but the cohomology of $P^n(C)$ is the polynomial algebra on A subject to the single relation $a^{n+1} = 0$ so the only possibility is that $m = n + 1$, and the projective bundle is $P^n(C) \longrightarrow P^n(C) \longrightarrow Y$

But Y is a manofold, so $\dim Y + n = n$, so $\dim Y = 0$, i.e. Y is a point (assuming that the manifolds we are considering are connected) This implies (with Cor.2 6) that $X = X'/P^n(C)$, so that we conclude $(P^n(C),X')$ can occur from blowing up (A,B) only in the two cases $(A,B) = (P^n(C),X')$ or $(A,B) = (X'/P^n(C),pt.)$.

3. Porteous' Theorem

The aim of this section is to prove the following theorem.

3.1 Theorem

Let X be a compact complex manifold, and $i : Y \longrightarrow X$ a compact complex submanifold. Let

$$\begin{tikzcd} Y' \arrow[r, "j"] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ Y \arrow[r, "i"'] & X \end{tikzcd}$$

be the result of blowing up Y. Then the equation

$$j_*(g^*E - L) = f^*TX - TX'$$

holds in $K(X')$ where E, L are the normal bundles of Y, Y' respectively.

3.2 In the case that X, Y are projective complex algebraic manifolds, this theorem is a cosequence of a similar theorem of Porteous ((10), Theorem 1, (iv))as follows. Let $\underline{A}_X$ denote the category of algebraic coherent sheaves on X, and let $\underline{L}_X \subset \underline{A}_X$ be the full subcategory of locally free sheaves. It is well known (3) that there is a natural equivalence betewen $\underline{L}_X$ and the category $\underline{V}_X$ of complex vector bundles over X, where the vector bundle E corresponds to the sheaf of germs of sections of E. Let $K^a(X)$ denote the algebraic Grothendieck group of $\underline{A}_X$ and $K(X)$ the topological group of $\underline{V}_X$. If $f : X \longrightarrow Y$ is a morphism then f induces a map $f^a : K^a(Y) \longrightarrow K^a(X)$ defined on a sheaf $F \in \underline{A}_Y$ by the equation

$$f^a(F) = \sum_p (-1)^p \operatorname{Tor}_p^{O_Y}(O_X, F)$$

where O_X, O_Y are the structure sheaves of X and Y. This construction makes K^a into a functor, and an important result in the theory is that there is a natural transformation $K^a \longrightarrow K$ whose construction depends on the fact that any $F \in \underline{A}_X$ has a finite resolution by objects in $\underline{L}_X$, i.e. by vector bundles.

The map $f : X \longrightarrow Y$ also induces a map $f^a_* : K^a(X) \longrightarrow K^a(Y)$ induced by taking the direct image of a sheaf. A theorem of

Atiyah-Hirzebruch (9) says that if $f:X \longrightarrow Y$ is an embedding, then the diagram

$$\begin{tikzcd} K^a(X) \arrow[r, "f^a_*"] \arrow[d] & K^a(Y) \arrow[d] \\ K(X) \arrow[r, "f_*"'] & K(Y) \end{tikzcd}$$

commutes, where f_* is the umkehr homomorphism. If F is a vector bunlde let $\underset{\sim}{F}$ denote the corresponding locally free sheaf. Porteous proves that

$$j^a_*(g^{a*}\underset{\sim}{E} - \underset{\sim}{L}) = f^{a*}\underset{\sim}{TX} - \underset{\sim}{TX'} \qquad \text{in } K^a(X')$$

which thus implies Theorem 3.1

The proof of Theorem 3.1 which we now intend to give holds for arbitrary compact complex manifolds and uses topological K-theory and does not make any use of the theory of coherent sheaves.

3.3 By differentiating the diagram

3.3.1

$$\begin{tikzcd} Y' \arrow[r, "j"] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ Y \arrow[r, "i"'] & X \end{tikzcd}$$

we obtain the commutative diagram

3.3.2

$$\begin{tikzcd} 0 \arrow[r] & TY' \arrow[r] \arrow[d, "Tg"'] & j^*TX' \arrow[r] \arrow[d, "Tf|Y"'] & L \arrow[r] \arrow[d, "N(f)"'] & 0 \\ 0 \arrow[r] & TY \arrow[r] & i^*TX \arrow[r] & E \arrow[r] & 0 \end{tikzcd}$$

Let $\emptyset = Tf:TX' \longrightarrow TX$. Since f is a diffeomorphism off Y', we see that $\emptyset$ is an isomorphism (of vector bundles) off Y'. Let U be a tubular neighbourhood of Y in $\hat{X}$, and consider the diagram

3.3.3

$$\begin{tikzcd} & K(X' \setminus U) & & \\ K(X',Y') \arrow[r] & K(X') \arrow[r] \arrow[u] & K(Y') \arrow[dr, "\lambda_L"] & \\ & K(X', X' \setminus U) \arrow[u] & K(\bar{U}, \partial U) \arrow[l, "k^{*-1}"] & K(BL,SL) \arrow[l, "q^{*-1}"] \end{tikzcd}$$

Now $g^*E - L \in K(Y')$ and what we intend to show is that the element

$$k^{*-1}q^{*-1}\lambda_L(g^*E - L) \in K(X', X' \setminus U)$$

is represented by the sequence

$$TX'|X' \setminus U \xrightarrow[\phi]{} f^*TX|X' \setminus U$$

which by Prop.1.4(iv) maps to $f^*TX - TX'$ in $K(X')$ which will prove Theorem 3.1 since j_* is by definition the composite map $K(Y') \longrightarrow K(X')$.

3.4 Restriction of the diagram 3.3.2 to vector bundles over Y' yields a commutative diagram

3.4.1

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & G \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & TY' \arrow[r] \arrow[d] & j^*TX' \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & g^*TY \arrow[r] \arrow[d] & j^*f^*TX \arrow[r] & g^*E \arrow[r] \arrow[d] & 0 \\
 & 0 & & G \otimes L \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

where G denotes the bundle along the fibres. By remarks in 1.7 and Prop.1.4(vii), the element $(g^*E - L).\lambda_L \in K(BL, SL)$ is represented by the sequence

3.4.2 $$(p^*g^*E \otimes p^*L^\vee) \oplus (p^*L \otimes \underline{1}) \xrightarrow[(1 \otimes e) \oplus (1 \otimes e^{-1})]{} (p^*g^*E \otimes \underline{1}) \oplus (p^*L \otimes p^*L^\vee)$$

We now choose (arbitrarily for the moment) splittings of the exact sequences in the diagram 3.4.1 as shown in the following diagram:

3.4.3

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & G \arrow[d, shift left, "y"] & & 0 \arrow[d, dash] & \\
0 \arrow[r] & TY' \arrow[u, dashed, shift left, "i"] \arrow[r, shift left, "a"] \arrow[d, shift left, "x"] & j^*TX' \arrow[l, dashed, shift left, "v"] \arrow[r, shift left, "b"] & L \arrow[l, dashed, shift left, "s"] \arrow[r] \arrow[d, shift left, "m"] & 0 \\
0 \arrow[r] & g^*TY \arrow[u, dashed, shift left, "j"] \arrow[r, shift left, "c"] \arrow[d] & j^*f^*TX \arrow[l, dashed, shift left, "t"] \arrow[r, shift left, "d"] & g^*E \arrow[l, dashed, shift left, "u"] \arrow[u, dashed, shift left, "r"] \arrow[r, dash] \arrow[d, shift left, "p"] & 0 \\
 & 0 & & G \otimes L \arrow[u, dashed, shift left, "n"] \arrow[d, dash] & \\
 & & & 0 &
\end{tikzcd}$$

we use the same letters to denote the maps induced between the corresponding bundles over BL.

3.5 In this number (3.5) we use the same letters to denote bundles over BL as the corresponding bundles over Y', for example G will denote either the bundle G over Y' along the fibres or the induced bundle p^*G over BL. However no confusion should arise since in this number we shall only be considering bundles over BL.

The idea of this section is to change the sequence (3.4.2) by using Proposition 1.4 and so represent the element $(g^*E - L).\lambda_L$ in a suitable form.

Let $\alpha : L \otimes L^\vee \longrightarrow \underline{1}$ be the evaluation isomorphism. From the diagram

3.5.1 $$\begin{tikzcd}
0 \arrow[r] & L \otimes L^\vee \arrow[r, shift left, "m \otimes 1"] & g^*E \otimes L^\vee \arrow[l, dashed, shift left, "r \otimes 1"] \arrow[r, shift left, "p \otimes 1"] & G \otimes L \otimes L^\vee \arrow[l, dashed, shift left, "n \otimes 1"] \arrow[r] \arrow[d, "1 \otimes \alpha"] & 0 \\
 & & & G \otimes 1 &
\end{tikzcd}$$

which yields a commutative diagram

$$\begin{tikzcd}
g^*E \otimes L \arrow[r, "1 \otimes e"] & g^*E \otimes \underline{1} \arrow[d, "\begin{pmatrix} r \\ p \end{pmatrix}"] \\
G \oplus (L \otimes L^{\vee}) \arrow[u, "((n \otimes 1)(1 \otimes \alpha^{-1}){,}\, m \otimes 1)"] \arrow[r, swap, "\begin{pmatrix} 0 & 1 \otimes e \\ (1 \otimes e)(1 \otimes \alpha^{-1}) & 0 \end{pmatrix}"] & L \oplus (G \otimes L)
\end{tikzcd}$$

we see that the element $(g^*E)\lambda_L$ is represented by the sequence

$$\begin{pmatrix} 0 & 1 \otimes e \\ (1 \otimes e)(1 \otimes \alpha^{-1}) & 0 \end{pmatrix} : G \oplus (L \otimes L^{\vee}) \longrightarrow L \oplus (G \otimes L)$$

(using Prop.1.4 (v)and (viii)).

From the commutative diagram (3.5.4) on the next page and using Prop.1.4 (v),(vii) and (viii), since the top sequence represents $(g^*E - L)\lambda_L$, so does the bottom sequence, Now using Prop.1.4 (v) and (vi), we see that $(g^*E - L)\lambda_L$ is represented by the sequence

<u>3.5.3</u>

$$(un(1 \otimes e)(1 \otimes \alpha^{-1})iv + cxv + umb) : j^*TX' \longrightarrow j^*f^*TX$$

(In diagram 3.5.4, the vertical maps except the two in the middle row are isomorphisms over BL and so do not affect the difference element. The two vertical maps in the middle row are inverse isomorphisms on SL, and so cancel out by Prop.1.4(vii).)

3.5.4

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
G \oplus (L \otimes L^{\vee}) \oplus L \oplus g^*TY \arrow[r, "{\begin{pmatrix} 0 & (1 \otimes e) & 0 & 0 \\ (1 \otimes e)(1 \otimes \alpha^{-1}) & 0 & 0 & 0 \\ 0 & 0 & (1 \otimes e^{-1}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}"] & L \oplus (G \otimes L) \oplus (L \otimes L^{\vee}) \oplus g^*TY \arrow[d, "{\begin{pmatrix} um & un & o & c \\ 0 & 0 & 1 & 0 \end{pmatrix}}"'] \\
j^*TX' \oplus (L \otimes L^{\vee}) \arrow[u, "{\begin{pmatrix} iv & 0 \\ 0 & 1 \\ b & 0 \\ xv & 0 \end{pmatrix}}"'] & j^*f^*TX \oplus (L \otimes L^{\vee}) \arrow[d, "{\begin{pmatrix} 1 & 0 \\ 0 & (1 \otimes e) \end{pmatrix}}"'] \\
j^*TX' \oplus L \arrow[u, "{\begin{pmatrix} 1 & 0 \\ 0 & (1 \otimes e^{-1}) \end{pmatrix}}"'] & j^*f^*TX \oplus L \arrow[d, "{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}"'] \\
j^*TX' \oplus L \arrow[u, "{\begin{pmatrix} av & s \\ b & 0 \end{pmatrix}}"'] \arrow[r, dashed, "{\begin{pmatrix} un(1 \otimes e)(1 \otimes \alpha^{-1})iv + cxv + umb & 0 \\ 0 & 1 \end{pmatrix}}"'] & j^*f^*TX \oplus L
\end{tikzcd}
$$

3.5.5 Let $\mu = (1 \otimes e)(1 \otimes \alpha^{-1}) : G \longrightarrow G \otimes L$.

3.6 Since $q: BL \longrightarrow U$ is homotopic to $BL \longrightarrow Y' \longrightarrow U$, we knnw that $j^*TX' \cong q^*(TX' U)$, and so the element

3.6.1 $$q^{*-1}((g^*E - L)\lambda_L) \in K(U, \partial U)$$

is represented by the sequence

3.6.2 $un\mu iv + cxv + umb : TX'|U \longrightarrow f^*TX|U$

where the map is an isomorphism on ∂U. It now remains to prove the assertion of 3.3, that the splittings can be chosen so that on ∂U this map is ϕ, the differential of $f: X' \longrightarrow X$. This map involves splittings u, n, i, v. In fact it is easier to show that the inverse of this map is the inverse of ϕ. Now on $BL = \partial U$ we have

$$\begin{aligned}(un\mu iv + cxv + umb)(ay\mu^{-1}pd + ajt + srd) &= unpd + ct + umrd \\ &= unpd + ct + u(1 - np)d \\ &= 1\end{aligned}$$

so $ay\mu^{-1}pd + ajt + srd$ is the inverse of $un\mu iv + cxv + umb$.

3.7 It is well known (10) that we can choose coordinate charts so that the map f is in a nice form, and we chhose splittings over these charts, and glue them together with partitions of unity:

We can choose charts $\{V\}$ of Y in X and $\{W\}$ of Y' in X' such that locally we have coordinate functions

$$x_1, \ldots, x_n : V \longrightarrow \mathbb{C}$$
$$x'_1, \ldots, x'_n : W \longrightarrow \mathbb{C}$$

with respect to which the map f has the form

$$\begin{aligned} x_i f &= x'_i && 1 \leqslant i \leqslant m \\ &= x'_i x'_n && m < i < n \\ &= x'_n && i = n. \end{aligned}$$

and where Y is given by the equations $x_j = 0$ ($m < j \leqslant n$)

and where Y' is given by the equation $x'_n = 0$.

3.8 Let $e_i = \frac{\partial}{\partial x_i}$ and $e'_i = \frac{\partial}{\partial x'_i}$. Then over W, the map $\emptyset$ is given by the equations

3.8.1
$$\begin{aligned}\emptyset(e'_i) &= e_i \quad (1 \leqslant i \leqslant m)\\ &= x'_n e_i \quad (m < i < n)\\ &= e_n + \sum_{m < i < n} x'_i e_i \quad \text{for } i = n.\end{aligned}$$

3.9 Now TY over W is spanned by e_i for $1 \leqslant i \leqslant m$, and TY' over W' is spanned by e'_i for $1 \leqslant i \leqslant n$. Define

3.9.1
$$\begin{aligned}t_W(e_i) &= e_i \quad \text{for} \quad 1 \leqslant i \leqslant m\\ &= 0 \quad \text{otherwise}\end{aligned}$$

3.9.2
$$j_W(e_i) = e'_i \quad \text{for} \quad 1 \leqslant i \leqslant m$$

3.9.3
$$\begin{aligned}v_W(e'_i) &= e'_i \quad \text{for} \quad 1 \leqslant i < n\\ &= 0 \quad \text{otherwise.}\end{aligned}$$

These splittings enable us to consider (over W)

3.9.4 E as spanned by e_i for $m < i \leqslant n$

G as spanned by e'_i for $m < i < n$

L as spanned by e'_n.

3.9.5 Hence $G \otimes L$ is spanned by $e'_i \otimes e_n$ for $m < i < n$.

With respect to this coordinate system, the map p is given by

3.9.6
$$\begin{aligned}p(e_i) &= x'_i\, e'_i \otimes e'_n \quad \text{for} \quad m < i < n\\ &= - \sum_{m < i < n} x'_i e'_i \otimes e'_n \quad \text{for } i = n\end{aligned}$$

and so we choose splittings

3.9.7
$$\begin{aligned}r_W(e_i) &= 0 \quad \text{for} \quad m < i < n\\ &= e'_n \quad \text{for } i = n\end{aligned}$$

3.9.8
$$n_W(e'_i \otimes e'_n) = e_i \qquad m < i \leqslant n.$$

3.10 Let φ_W be a partition of unity subordinate to the open cover $\{W\}$ of Y'.

3.11 We consider now the map $\mu : G \longrightarrow G \otimes L$. Over W we have a section of L given by e'_n. Let ε_n be the dual section of L, i.e.

$$\text{3.11.1} \qquad \langle \varepsilon_n, e'_n \rangle = 1 .$$

Then $\alpha^{-1} : \underline{1} \longrightarrow L \otimes L^{\vee}$ is defined by the section

$$\sum_W \varphi_W \, e'_n \otimes \varepsilon_n .$$

So if $g \in \pi^* G$ lies over the point $1 \in BL$, then

$$\mu(g) = (1 \otimes e)\left(\sum_W \varphi_W \, g \otimes e'_n \otimes \varepsilon_n \right)$$

$$= \sum_W \varphi_W g \otimes e'_n \langle \varepsilon_n, 1 \rangle$$

Now we can write $1 = ze'_n$ for some $z \in \mathbb{C}$, then

$$\langle \varepsilon_n, 1 \rangle = \langle \varepsilon_n, ze'_n \rangle = z \text{ , whence}$$

$$\text{3.11.2} \quad \mu(g) = \sum_W \varphi_W \, g \otimes 1 = g \otimes 1$$

In particular, over the point in U with coordinates $(x'_1, \dots, x'_n)$ (which we are identifying with the vector $x'_n e'_n$ of L) the map is given by

$$\text{3.11.3} \qquad \mu(e'_i) = e'_i \otimes x'_n e'_n = x'_n \, e'_i \otimes e'_n .$$

Thus away from the zero section (i.e. where $x'_n = 0$) we have

$$\text{3.11.4} \qquad \mu^{-1}(e'_i \otimes e'_n) = \frac{1}{x'_n} \, e'_i .$$

3.12 Finally, define splitting

$$s_W(e'_n) = e'_n$$

3.13 Lemma Over W we have

$$ay \, \mu^{-1} pd + aj_W t_W + s_W r_W d = \emptyset^{-1}$$

Proof: We check equality on each e_i for $1 \leqslant i \leqslant n$.

(i) $1 \leqslant i \leqslant m$. It is easily seen that each side maps e_i to e'_i.

(ii) $m < i < n$. Now $aj_W t_W(e_i) = 0$ by 3.9.1, and $s_W r_W d(e_i) = 0$ by 3.9.7.

$$\begin{aligned} ay \, \mu^{-1} pd(e_i) &= ay \, \mu^{-1}(e'_i \otimes e'_n) \quad \text{by 3.9.6} \\ &= ay((1/x'_n) \, e'_i) \\ &= (1/x'_n) \, e'_i . \end{aligned}$$

$$\begin{aligned} \text{Thus } (ay\mu^{-1}pd + aj_W t_W + s_W r_W d)(e_i) &= (1/x'_n)(e'_i) \\ &= \emptyset^{-1}(e_i) \text{ by 3.8.1.} \end{aligned}$$

(iii) $i = n$. Now $aj_W t_W(e_n) = 0$ by 3.9.1.

$$ay\mu^{-1}pd(e_n) = ay\mu^{-1}\Big(-\sum_{m<i<n} x_i'e_i' \otimes e_n'\Big) \quad \text{by 3.9.6}$$

$$= ay\Big(-\sum_{m<i<n} (x_i'/x_n')\, e_i'\Big) \quad \text{by 3.11.4.}$$

$$= -\sum_{m<i<n} (x_i'/x_n')\, e_i'$$

And $s_W r_W d(e\ ) = e_n'$ by 3.9.7 and 3.12.

Hence

$$(ay\mu^{-1}pd + aj_W t_W + s_W r_W d)(e_n) = e_n' - \sum_{m<i<n} (x_i'/x_n')\, e_i' \,.$$

But from 3.8.1,

$$e_n' = \phi^{-1}(e_n) + \phi^{-1}\Big(\sum_{m<i<n} x_i'e_i\Big)$$

$$= \phi^{-1}(e_n) + \sum_{m<i<n} (x_i'/x_n')e_i'$$

whence

$$(ay\mu^{-1}pd + aj_W t_W + s_W r_W d)(e_n) = \phi^{-1}(e_n)$$

////.

3.14 We are now ready to define the global splittings j,t,s,r. Let $h_W = j_W t_W$ and $k_W = s_W r_W$. Let $h = \sum_W \varphi_W h_W$ and $k = \sum_W \varphi_W k_W$.

Now since $xh_W c = 1$, we know that $xhc = 1$, and similarly $bkw = 1$. Put $j = hc$, $t = xh$, $r = bk$, and $s = km$. Then $xj = 1 = tc$, and $bs = 1 = rm$, so j,t,s,r are .

3.15 Multiplying the equation of Lemma 3.13 by φ_W, and adding ov r $\{W\}$ gives

$$ay\mu^{-1}pd + ah + kd = \phi^{-1}.$$

Now from $cxj_W t_W = 1 = cxh$, and similarly $mbk = 1$, we have

$$ay\mu^{-1}pd + ahcxh + kmbkd = \phi^{-1}$$

i.e. $ay\mu^{-1}pd + ajt + srd = \phi^{-1}$ as required, which completes the proof of Theorem 3.1.

3.16 Corollary Let $v = c_1(L)$

(i) The behaviour of Chern character under blowing up is expressed by the formula

$$f^*ch(X) - ch(X') = j_*\left[(g^*ch(E) - e^v)\left(\frac{1 - e^{-v}}{v}\right)\right]$$

(ii) The behaviour of Chern classes under blowing up is expressed by the formula

$$f^*C(X) - C(X^*) = j_*\left[\left(\frac{1}{v}\right)\frac{g^*C(E)}{(1+v)\sum_i (1-v)^i . g^*c_{n-m-i}(E)}\right]$$

in $\hat{G}(X')$, or equivalently

$$C(X') = f^*C(X).\left(1 + j_*\left[\left(\frac{1}{v}\right)\frac{(1+v)\sum_i (1-v)^i . g^*c_{n-m-i}(E)}{g^*C(E)} - 1\right]\right)$$

in $H^*(X',\mathbb{Z})$.

Proof: (i) From Theorem 3.1,

$$\begin{aligned} f^*ch(X) - ch(X') &= ch\, j_*(g^*E - L) \\ &= j_*\left[(g^*ch(E) - e^v).Todd(L)\right] \quad \text{By 10.4} \\ &= j_*\left[(g^*ch(E) - e^v)((1 - e^{-v})/v)\right]. \end{aligned}$$

(ii) Similarly

3.16.1 $f^*C(X) - C(X') = Cj_*(g^*E - L)$.

Now $g^*E - L = G \otimes L = F$ say. Then by 10.6

3.16.2 $Cj_*(F) = j_*((\frac{1}{v})C(F.\, \lambda_{-1}\check{L}))$

Now $C(F.\, \lambda_{-1}L) = C(g^*E.\, \lambda_{-1}\check{L}) - C(L.\, \lambda_{-1}L)$

$$= \frac{g^*C(E)}{\sum_i (1-v)^i g^*c_{n-m-i}(E)} - \frac{1+v}{v + (1-v)}$$

3.16.3
$$= \frac{g^*C(E)}{(1+v)\sum_i (1-v)^i g^*c_{n-m-i}(E)}$$

Formula (a) now follows from 3.16.1, 3.16.2 and 3.16.3.

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CHAPTER 3 :

SUSPENDING & COLLAPSING CHERN CLASSES

SUSPENDING AND COLLAPSING CHERN CLASSES

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Introduction

If X, Y are (based) topological spaces, let $[X,Y]_*$ denote the set of homotopy classes of based maps from X to Y, and let $[X,Y]$ denote the set of homotopy classes of maps from X to Y.

If ξ is a principal G bundle over SX, then it is classified by an element of $[SX, B_G]$ where B_G denotes the classifying space of the topological group G. Conversely, if we take trivialisations of ξ over the two cones of SX, these differ on their intersection by a map $X \longrightarrow G$, and ξ is classified by the corresponding element of $[X,G]$. By considering the classifying bundle

$$G \longrightarrow E_G \longrightarrow B_G$$

it is well known that there is an isomorphism

$$[X,G]_* \cong [SX,B_G]_*$$

The chapter starts with a theorem which implies for a compact Lie group G that

$$[X,G] \cong [SX,B_G]$$

and by using Milnor's construction of B_G, we prove in §3 that the two elements which classify ξ correspond under this isomorphism.

The case in which we are interested is $G = U(n)$, and we use the theorem in this case to determine the Chern classes of a vector bundle over SX in terms of its classifying element in $[X,U(n)]$. For this we use the well known results on the cohomology of $U(n)$ and $B_{U(n)}$, and obtain the following proposition:

4.2 Proposition *If ξ is a principal $U(n)$ bundle over SX, and trivialisations over the cones differ by a map $\varphi: X \longrightarrow U(n)$, then*

$$c_i(\xi) = \sigma^{-1}\varphi^*(x_i)$$

where $\sigma: H^{2i}(SX,Z) \longrightarrow H^{2i-1}(X,Z)$ is the suspension isomorphism, and x_i is a (certain) generator of $H^{2i-1}(U(n),Z)$.

In section 5, we consider a vector bundle over X which is trivial over the subspace A, and examine the effect on Chern classes of collapsing the bundle in two different ways. The main result of this section is the following:

<u>5.2 Theorem</u> <u>If</u> (X,A) <u>has the homotopy extension property, and</u> E <u>is a vector bundle over</u> X, <u>and</u> α β <u>are trivialisations of</u> E <u>over</u> A <u>which thus differ by a map</u> $f:A \longrightarrow Gl_n(\mathbb{C})$, <u>then</u>

$$c_i(E/\alpha) - c_i(E/\beta) = \delta f^* x_i'$$

<u>where</u> $\delta : H^{2i-1}(X) \longrightarrow H^{2i}(X/A)$ <u>is the coboundary, and</u> x_i' <u>is a (certain) generator of</u> $H^{2i-1}(Gl_n(\mathbb{C}),Z)$.

The proof of this theorem makes use of the previous results on bundles over suspensions.

In section 6 we give some examples of the theorem, of which one is the classical result concerning Chern classes of vector bundles over spheres, of which this chapter can be thought to be a generalisation.

Theorem 1.5 and Proposition 5.6 are probably well known, although no reference to them could be found.

1 The fundamental isomorphism

1.1 Let $p:E \longrightarrow B$ be a fibre space with fibre F, and with E, F and B path connected paracompact hausdorff spaces. Then p is a fibration (Spanier, [1] Ch.2, 7.14). For $b \in B$, let F_b denote the fibre over b. It is well known that for any space X, the sets

$$\{ [X,F_b] \; : b \in B \}$$

form a local system over B, where $[X,F_b]$ denotes the set of homotopy classes of maps from X to F_b. For suppose $\omega:I \longrightarrow B$ is a path in B from b to c, and $g:X \longrightarrow F_b$ represents an element 'g' in $[X,F_b]$. Consider the diagram

1.2
$$\begin{tikzcd} E \arrow[rr] & & B \\ & & I \arrow[u, "\omega"'] \\ X \times 0 \arrow[uu, dashed, "g"] \arrow[rr, dashed] & & X \times I \arrow[u, "\text{projection}"'] \arrow[uull, dashed, "G"'] \end{tikzcd}$$

Since p is a fibration, there exists a map $G:X \times I \longrightarrow E$ making the diagram commute. The map $G_1:X \longrightarrow E$ defined by $G_1(x) = G(x,1)$ maps into F_c and so defines an element $[G_1] \in [X,F_c]$. This homotopy class is well defined and depends only on the homotopy class of $g:X \longrightarrow F_b$, so this construction defines a map

1.3
$$\omega_* : [X,F_b] \longrightarrow [X,F_c]$$

which defines the local system.

1.4 Throughout sections 1 and 2, let $p:E \longrightarrow B$ be a fibre space with fibre F such that

(i) E, F and B are path connected paracompact hausdorff spaces

(ii) E is contractible to a point

(iii) B has a base point $*$ and $\pi_1(B,*) = 0$

Let $e:E \times I \longrightarrow E$ be a contraction with e_1 = identity, and $e_0(E) = e$ for some point $e \in E$. We note that by condition (iii) any local system on B is simple. Let F_* denote the fibre over the base point.

1.5 Theorem

There is a natural isomorphism

$$[X, F_*] \xrightarrow{\cong} [SX, B]$$

Proof: Suppose $g: X \longrightarrow F_*$ represents an element $[g]$ in $[X, F_*]$. Then

$$pe(g \times 1): X \times I \longrightarrow E \times I \longrightarrow E \longrightarrow B$$

respects the identification map $X \times I \longrightarrow SX$ and so induces a map $\hat{g}: SX \longrightarrow B$:

$$\begin{tikzcd} X \times I \arrow[r, "pe(g \times 1)"] \arrow[d] & B \\ SX \arrow[ur, dashed, "\hat{g}"'] & \end{tikzcd}$$

The homotopy class of $\hat{g}$ depends only on the homotopy class of g. For suppose that $G: g \simeq g': X \longrightarrow F_*$. Then the map from $X \times I \times I$ to B given by

$$(x, t, u) \longmapsto pe(G(x, u), t)$$

respects the map $X \times I \times I \longrightarrow SX \times I$, which is again an identification and induces a homotopy $\hat{G}: \hat{g} \simeq \hat{g}': SX \longrightarrow B$

$$\begin{tikzcd} X \times I \times I \arrow[r, dotted] \arrow[d] & B \\ SX \times I \arrow[ur, dashed, "\hat{G}"'] & \end{tikzcd}$$

This construction thus defines a function

$$\emptyset: [X, F_*] \longrightarrow [SX, B]$$

which is clearly natural. In order to show that $\emptyset$ is an isomorphism we construct the inverse.

Suppose $h: SX \longrightarrow B$ represents some element $[h]$ in $[SX, B]$. Let $\underline{0}$ and $\underline{1}$ denote the images in SX of the points $(x, 0)$ and $(x, 1)$ from $X \times I$. Choose some point $a \in F_{h(\underline{0})}$. Consider the diagram:

$$\begin{tikzcd} E \arrow[rr, "p"] & & B \\ & & SX \arrow[u, "h"'] \\ X \times \{0\} \arrow[u, "\text{constant map to } a"] \arrow[uu] \arrow[rr, dotted] & & X \times I \arrow[u] \arrow[uull, dashed, "H"'] \end{tikzcd}$$

By the homotopy lifting property, there exists a map $H:X \times I \longrightarrow E$ making the diagram commute, and moreover any two such are fibre homotopic. We say that H lifts h based at a. Then $H_1:X \longrightarrow F_{h(\underline{1})}$ defines an element of $[X,F_{h(\underline{1})}]$ which is independent of the choice of lifting. Since $\pi_1(B) = 0$, the local system $\{[X,F_b] : b \in B\}$ is simple, so there is a natural isomorphism

$$[X,F_{h(\underline{1})}] \longrightarrow [X,F_*]$$

which is induced (see 1.3) by <u>any</u> path in E from h(1) to the base point. Hence corresponding to h and a there is an element in $[X,F_*]$. We show firstly that this element does not depend on the choice of a, and secondly that it does not depend on the choice of h representing the element $[h]$ in $[SX,B]$ so that this const uction defines a function

$$\psi: [SX,B] \longrightarrow [X,F_*]$$

First suppose that a' is a second point in $F_{h(\underline{0})}$. Let $\omega:I \longrightarrow E$ be a path from a to a'. Consider the diagram:

$$\begin{tikzcd}
E \arrow[rrr, "p"] & & & B & \\
 & & & SX \arrow[u, "h"'] & {[x,t]} \\
X \times \{0\} \times I \arrow[uu, "\omega"] \arrow[rrr] & & & X \times I \times I \arrow[u] \arrow[ulll, dashed, "\Omega"'] & (x,t,u) \arrow[u, mapsto]
\end{tikzcd}$$

By the homotopy lifting property, there exists a map

$$\Omega:X \times I \times I \longrightarrow E$$

making the diagram commute. Then

$\Omega(-,1,0):X \longrightarrow E$ lifts h based at a and

$\Omega(-,1,1):X \longrightarrow E$ lifts h based at a'.

Moreover $\Omega(x,1,u) \in F_{h(\underline{1})}$ for all $u \in I$, so that

$$\Omega(-,1,-):X \times I \longrightarrow F_{h(\underline{1})}$$

defines a homotopy between the two choices of maps from X to $F_{h(\underline{1})}$ so that the element in $[X,F_*]$ is independent of the choice of the point in the fibre.

Secondly, suppose that $H:h \simeq h':SX \longrightarrow B$. Choose points $c \in F_{h(\underline{0})}$ and $c' \in F_{h'(\underline{0})}$ and let $\omega:I \longrightarrow E$ be a path from c to c', which covers the path $H(\underline{0},t)$ in B from $h(\underline{0})$ to $h'(\underline{0})$

$$\begin{tikzcd}
E \arrow[rr, "p"] & & B \\
& & SX \times I \arrow[u, "H"'] \\
X \times \{0\} \times I \arrow[uu, "\omega"] \arrow[rr, hook] & & X \times I \times I \arrow[u] \arrow[uull, dashed, "J"']
\end{tikzcd}$$

By the homotopy lifting property, there exists a map $J:X \times I \times I \longrightarrow E$ making the diagram commute. Then

$J(-,1,0):X \longrightarrow F_{h(\underline{1})}$ lifts h based at c

$J(-,1,1):X \longrightarrow F_{h'(\underline{1})}$ lifts h' based at c'.

Now as u varies, $pJ(x,1,u) = H(\underline{1},u)$ defines a path in B from $h(\underline{1})$ to $h'(\underline{1})$, and the isomomrphism

$$[X,F_{h(\underline{1})}] \rightarrow [X,F_{h'(1)}]$$

induced by this path is easily seen (from the following diagram and compare 1.2) to take the class of $J(-,1,0)$ to the class of $J(-,1,1)$:

$$\begin{tikzcd}
E \arrow[r, "p"] & B & & H(1,u) \\
X \times \{0\} \arrow[u, "{J(-,1,0)}"] \arrow[r, hook] & X \times I \arrow[u] \arrow[ul, dashed, "{J(-,1,-)}"'] & & (x,u) \arrow[u, mapsto]
\end{tikzcd}$$

Again since the local system is simple, the two liftings define the same element in $[X,F_*]$, so that

$$\Psi:[SX,B] \rightarrow [X,F_*]$$

is well defined.

It is easile seen that $\Psi\psi = 1$. To show that $\Phi\Psi = 1$, we observe that $\underline{0} \cup \underline{1} \subset SX$ has the homotopy extension property, so that any element in $[SX,B]$ may be represented by a map $h:SX \longrightarrow B$ such that $h(\underline{0}) = p(e)$ and $h(1) = *$. From the diagram

$$\begin{tikzcd}
E \arrow[rr, "p"] & & B \\
& & SX \arrow[u, "h"'] \\
X \times \{0\} \arrow[uu, "\text{constant map to } e"] \arrow[rr, hook] & & X \times I \arrow[u] \arrow[uull, dashed, "H"']
\end{tikzcd}$$

let H_1 represent $\Psi[h]$. To prove that $\phi\Psi[h] = [h]$, we must show that $\quad pe(H_1 \times 1) \simeq pH : X \times I \longrightarrow B \text{ rel } X \times \partial I$ since $pe(H_1 \times 1)$ induces a map on SX representing $\phi\Psi[h]$, and pH induces h on SX. It suffices to show that

$$e(H_1 \times 1) \simeq H : X \times I \longrightarrow E \text{ rel } X \times \partial I.$$

We observe that

$$e(H_1 \times 1)(x,t) = e(H(x,1),t)$$

while

$$H(x,t) = e(H(x,t),1)$$

The result then follows from the next lemma.

<u>1.6 Lemma</u>

<u>There exists a map</u> $r : I \times I \longrightarrow I \times I$ <u>such that</u>

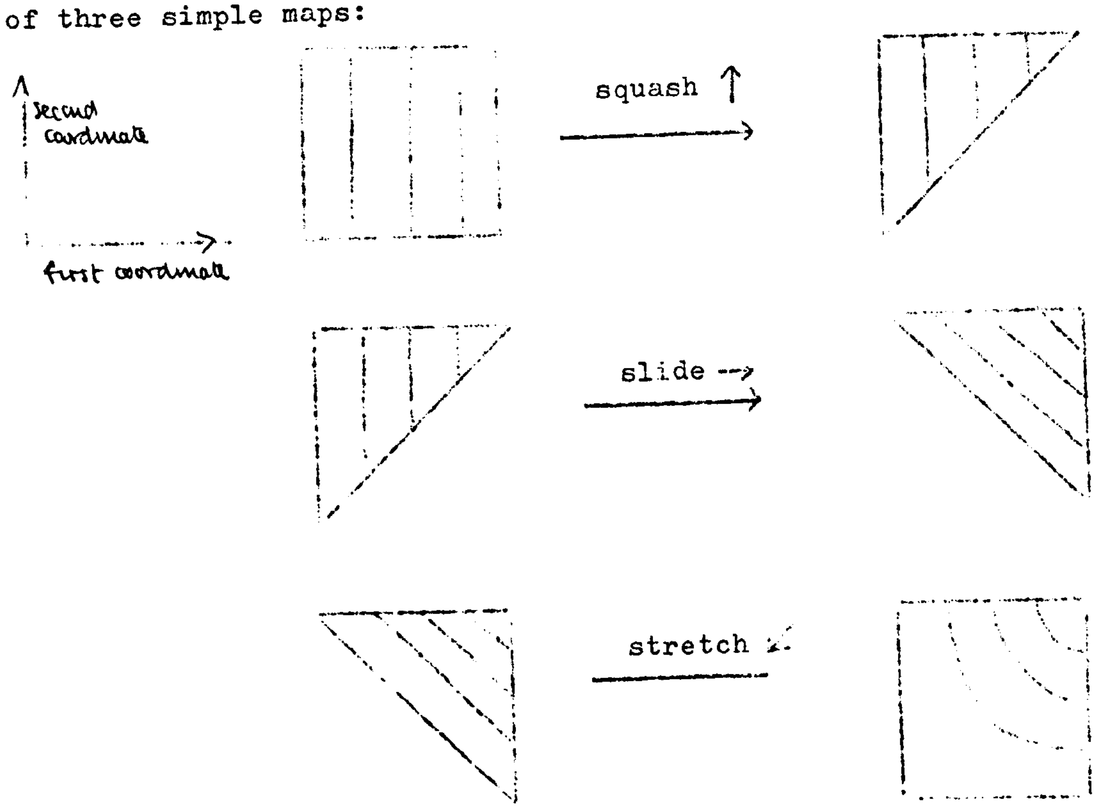
(i) $r \mid I \times \{1\}$ <u>is the identity</u>

(ii) $r(t,0) = (1,t)$

(iii) $r(\{1\} \times I) = (1,1)$

(iv) $r(\{0\} \times I) = \{0\} \times I \cup I \times \{0\}$

<u>Proof</u>: We illustrate the existence of such a map as the composite of three simple maps:



Let $r:I \times I \longrightarrow I \times I$ be a map satisfying the conditions of lemma 1.6. Write $r(t,u) = (r_1(t,u), r_2(t,u))$. Now define $F:X \times I \times I \longrightarrow E$ by

$$F(x,t,u) = e(H(x,r_1(t,u), r_2(t,u)).$$

This is the required homotopy, which completes the proof of Theorem 1.5.

////.

2. The cohomology isomorphism

2.1 For $n \geqslant 1$, we have the following diagram

2.2
$$\begin{tikzcd} & & & \tilde{H}^{n-1}(E) \arrow[d] \\ H^n(B) & H^n(B,*) \arrow[l, "i^*"', "\cong"] \arrow[r, "p^*"] & H^n(E,F_*) \arrow[d] & H^{n-1}(F_*) \arrow[l, "\delta"'] \\ & & H^n(E) & \end{tikzcd}$$

Since E is contractible, $\tilde{H}^*(E) = 0$, so δ is an isomorphism. Let

2.3
$$\sigma = \delta^{-1}p^*(i^*)^{-1} : H^n(B) \longrightarrow H^{n-1}(F_*)$$

be the composite homomorphism.

2.4 Proposition

If $h:SX \longrightarrow B$ and $g:X \longrightarrow F_*$ are such that the homotopy classes they define correspond under the isomorphism of Theorem 1.5, then the following diagram commutes:

$$\begin{tikzcd} H^n(B) \arrow[r, "\sigma"] \arrow[d, dashed, "h^*"'] & H^{n-1}(F_*) \arrow[d, "g^*"] \\ H^n(SX) \arrow[r, "\sigma"] & H^{n-1}(X) \end{tikzcd}$$

where $\sigma:H^n(SX) \longrightarrow H^{n-1}(X)$ is the suspension isomorphism.

Proof: We may assume without loss of generality (as in Theorem 1.5) that $h(0) = p(e)$ and $h(1) = *$. Let $H:X \times I \longrightarrow E$ lift h (as in 1.5.7). Then H_1 is homotopic to g, so $g^* = H_1{}^*$.

Let CX be the cone

$$CX = \frac{X \times I}{X \times \{0\}}$$

Then $H:X \times I \longrightarrow E$ facotrs through CX by a map (also denoted by H) and moreover if we consider $X \subset CX$ as $X \times \{1\}$, then

$$H_1 = H|X : X \longrightarrow F_*$$

From the commutative diagram

2.4.1

$$\begin{tikzcd}
E \arrow[rr, "p"] & & B \\
& & SX \arrow[u, "h"'] \\
& & CX \arrow[u] \arrow[ull, dashed] \\
X \times \{0\} \arrow[uuu, "\text{constant map to } e"] \arrow[rr, hook] & & X \times I \arrow[u] \arrow[uuull]
\end{tikzcd}$$

we obtain the commutative diagram:

2.4.2

$$\begin{tikzcd}
B \arrow[r] & (B,*) & (E,F_*) \arrow[l, "p"'] & F_* \arrow[l] \\
SX \arrow[u, "h"] \arrow[r, dash] & (SX,\underline{1}) \arrow[u, "h"] & (CX,X) \arrow[l, "c"] \arrow[u, "H"] & X \arrow[l] \arrow[u, "H_1"]
\end{tikzcd}$$

where $c:(CX,X) \longrightarrow (SX,\underline{1})$ collapses X to $\underline{0}$. This yields a commutative diagram

$$\begin{tikzcd}
H^n(B) & H^n(B,*) \arrow[l] \arrow[r, "p^*"] \arrow[d, "h^*"] & H^n(E,F_*) \arrow[d, "H^*"] & H^{n-1}(F_*) \arrow[l, "\delta"'] \arrow[d, "H_1{}^*"] \\
H^n(SX) \arrow[from=u, "h^*"] & H^n(SX,\underline{1}) \arrow[l] \arrow[r, "c^*"'] & H^n(CX,X) & H^{n-1}(X) \arrow[l, "\delta"]
\end{tikzcd}$$

By definition, the top row is $\sigma : H^n(X) \longrightarrow H^{n-1}(F_*)$ and the bottom row is the suspension isomorphism $\sigma : H^n(SX) \longrightarrow H^{n-1}(X)$, thus proving the proposition.

////.

3. Milnor's construction

In this section we describe Milnor's construction of the classifying bundle associated with a topological group based on the exposition given in (2).

3.1 Let G be a topological group, and let E'_G denote the set of sequences of pairs

3.1.1 $((t_0,x_0),(t_1,x_1),\dots)$

where $x_i \in G$ and $t_i \in [0,1]$ such that only a finite number of $t_i \neq 0$ and $\sum_i t_i = 1$. We define an equivalence relation on E'_G by

3.1.2 $((t_0,x_0),(t_1,x_1),\dots) \sim ((t'_0,x'_0),(t'_1,x'_1),\dots)$

if $t_i = t'_i$ for all i, and $x_i = x'_i$ for those values of i for which $t_i \neq 0$.

Let E_G be the set of equivalence classes. We denote the equivalence class of

$$((t_0,x_0),(t_1,x_1),\dots)$$

by

3.1.3 $(t_0x_0,t_1x_1,\dots\dots)$.

For each i, define

3.1.4 $t_i:E_G \longrightarrow [0,1]$ by $(t_0x_0,t_1x_1,\dots) \longmapsto t_i$

and then define

3.1.5 $x_i:t_i^{-1}(0,1] \longrightarrow G$ by $(t_0x_0,t_1x_1,\dots) \longmapsto x_i$

We give E_G the smallest topology such that the functions t_i,x_i are continuous for all $i \geq 0$.

The map $E_G \times G \longrightarrow E_G$ defined by

$$(t_0x_0,t_1x_1,\dots) \times g \longmapsto (t_0x_0g,t_1x_1g,\dots)$$

defines a right action of G, and moreover this map is continuous.

Let B_G be the orbit space (with the quotient topology.) Then $G \longrightarrow E_G \longrightarrow B_G$ is a principal G bundle, and is a classifying bundle for principal G bundles over, say, paracompact spaces. Since we shall need an explicit descr ption of a classifying map, we describe its construction, again based on (2).

3.2 Suppose X is a paracompact space, and $G \longrightarrow E \xrightarrow{p} X$ is a principal G bundle. Let $\{U_i : i = 0,1,\dots\}$ be an open cover of X with the property that $E|U_i$ is trivial, and let

$$\{f_i : X \longrightarrow [0,1] \quad i = 0,1,\dots\}$$

be a partition of unity subordinate to the cover $\{U_i\}$.

Let $h_i : U_i \times G \longrightarrow E|U_i$ be a trivialisation, and let

$q_i : U_i \times G \longrightarrow G$ be projection.

Then the map $h : E \longrightarrow E_G$ defined by

$$h(e) = (f_0 p(e) q_0 h_0^{-1}(e),\ f_1 p(e) q_1 h_1^{-1}(e),\ \dots)$$

is a G map, so induces a map $X \longrightarrow B_G$ and this map classifies E.

3.3 Let 1 denote the identity in G, Then if we denote the element of B_G determined by the element $(t_0x_0, t_1x_1, \dots)$ by

$$[t_0x_0, t_1x_1, \dots]$$

then we can give B_G the base point

$$[11, 0, 0, \dots]$$

(where '0' abbreviates '0g' for some $g \in G$)

so that G can be naturally identified with the fibre over the base point.

3.4 Suppose $G \longrightarrow E \longrightarrow SX$ is a principal G bundle, with X,G paracompact. Since the upper and lower half cones of SX are contractible, this bundle is trivial over them. If we choose a trivialisation over each half, and compare them on the intersection of the two cones , i.e. on X, they differ by a map $X \longrightarrow G$.

Suppose now that G is a compact connected semisimple Lie group. Then since $\pi_2(G) = 0$ (see for example (3)) and since E_G is contractible, we see from the exact homotopy sequence of the fibration $G \longrightarrow E_G \longrightarrow B_G$ that

$\pi_1(B_G) = \pi_2(G) = 0$, so that the fibre bundle

$$G \longrightarrow E_G \longrightarrow B_G$$

is the type we have been considering.

In particular, by Theorem 1.5, the map $X \longrightarrow G$ described above determines a map $SX \longrightarrow B_G$. We now show that this map classifies the bundle $G \longrightarrow E \longrightarrow SX$.

3.5 Proposition

Let G be a Lie group, and $G \longrightarrow E \longrightarrow SX$ a principal G bundle, where X is paracompact. Then trivialisations over the cones differ by a map $X \longrightarrow G$ which corresponds under the isomorphism of Theorem 1.5 to a classifying map for the bundle.

Proof: Let C_0, C_1 be the open cones

$$C_0 = \{[x,t] \ : \ 0 \leqslant t < 2/3\}$$
$$C_1 = \{[x,t] \ : \ 1/3 < t \leqslant 1\}$$

and let

$$h_i : C_i \times G \longrightarrow E|C_i \quad \text{for } i = 0,1 \quad \text{be trivialisation.}$$

Then on $X = \{[x,\tfrac{1}{2}] : x \in X\} \subset SX$, these trivialisations differ by a map $\varphi : X \longrightarrow G$, i.e.

$$h_0([x,\tfrac{1}{2}], g) = h_1([x,\tfrac{1}{2}], \varphi(g)).$$

Let $f_i : X \longrightarrow [0,1]$ be a partition of unity subordinate ti this open cover, and let $q_i : C_i \times G \longrightarrow G$ be projection. Then by 3.2, the map $h : E \longrightarrow E_G$ given by

$$h(e) = (f_1 p(e) q_1 h_1^{-1}(e), f_0 p(e) q_0 h_0^{-1}(e), 0, 0, \ldots)$$

induces a map $SX \xrightarrow{\hat{h}} B_G$ which classifies E. This map can be described as follows: if $[x,t] \in SX$, then

$$\hat{h}([x,t]) = [f_1 p(e) q_1 h_1^{-1}(e), f_0 p(e) q_0 h_0^{-1}(e), 0, 0, \ldots]$$

where e is some point in the fibre over $[x,t]$. We now show that under the isomorphism of Theorem 1.5, this map corresponds to φ.

Now over C_0 the bundle E is trivial and there is a section

$$[x,t] \longrightarrow h_0([x,t], 1)$$

so that over C_0, the map $\hat{h}$ can be described by

$$\begin{aligned}
\hat{h}(x,t) &= [f_1[x,t]q_1h_1^{-1}h_0([x,t], 1), f_0[x,t]q_0h_0^{-1}h_0([x,t], 1),0,0,\ldots] \\
&= [f_1[x,t]q_1h_1^{-1}h_0([x,t], 1), f_0[x,t]1,0,0,0,\ldots]
\end{aligned}$$

Define $n: X \times [0,\frac{1}{2}] \longrightarrow E_G$ to be

$$n(x,t) = (f_1[x,t]q_1h_1^{-1}h_0([x,t], 1), f_0[x,t]1,0,0,0,\ldots)$$

Then the following diagram commutes:

$$\begin{tikzcd}
E_G \arrow[rr] & & B_G \\
& & C_0 \arrow[u, "\hat{h}"'] \\
X \times \{0\} \arrow[uu, "\begin{array}{c}\text{constant}\\ \text{map to}\\ (0,11,0,0,..)\end{array}"] \arrow[rr, leftrightarrow] & & X \times [0,\frac{1}{2}] \arrow[u] \arrow[uull, "n"]
\end{tikzcd}$$

Similarly, over C_1 there is a section

$$[x,t] \longmapsto h_1([x,t], \varphi(x))$$

so that over C_1 the map $\hat{h}$ can be descreibed by

$$\hat{h}([x,t]) = [f_1[x,t]\varphi(x), f_0q_0h_0^{-1}h_1([x,t], \varphi(x)),0,0,\ldots]$$

Define $m: X \times [\frac{1}{2},1] \longrightarrow E_G$ to be

$$m(x,t) = (f_1[x,t]\varphi(x), f_0q_0h_0^{-1}h_1([x,t], \varphi(x)),0,0,\ldots)$$

Then the following diagram commutes:

$$\begin{tikzcd}
E_G \arrow[rr] & & B_G \\
& & C_1 \arrow[u, "\hat{h}"'] \\
& & X \times [\frac{1}{2},1] \arrow[u] \arrow[uull, "m"]
\end{tikzcd}$$

$$\begin{aligned}
\text{Now}\quad n(x,\tfrac{1}{2}) &= (f_1[x,\tfrac{1}{2}]q_1h_1^{-1}h_0([x,\tfrac{1}{2}], 1), f_0[x,\tfrac{1}{2}]1,0,0,\ldots) \\
&= (f_1[x,\tfrac{1}{2}]\varphi(x), f_0[x,\tfrac{1}{2}]1,0,0,\ldots) \\
\text{and}\quad m(x,\tfrac{1}{2}) &= (f_1[x,\tfrac{1}{2}]\varphi(x), f_0[x,\tfrac{1}{2}]q_0h_0^{-1}h_1([x,\tfrac{1}{2}], \varphi(x)),0,0,\ldots) \\
&= (f_1[x,\tfrac{1}{2}]\varphi(x), f_0[x,\tfrac{1}{2}]1,0,0,\ldots)
\end{aligned}$$

so that n and m agree on $X \times \{\frac{1}{2}\}$ so combine to give a map from $X \times I$ to E_G such that the following diagram commutes:

$$\begin{tikzcd}
E_G \arrow[r, dashed] & B_G \\
X \times \{0\} \arrow[u, "\text{constant map to } (0,11,0,0,\ldots)"] \arrow[r] & X \times I \arrow[u, "\hat{h}"'] \arrow[ul, dashed]
\end{tikzcd}$$

and so by Theorem 1.5 the map from $X = X \times \{1\}$ to E_G corresponds to the classifying map $\hat{h}$. But this map is m_1, that is

$$m_1(x) = m(x,1) = (f_1[x,1]\varphi(x), f_0[x,1]q_0h_0^{-1}h_1([x,1],\varphi(x)),0,0,\ldots)$$

$$= (1\varphi(x),0,0,\ldots.)$$

i.e. $m_1 = \varphi$, which completes the proof of the proposition.

////.

4. Vector bundles over suspensions

It is well known (see for example Browder (4)) that the cohomology ring $H^*(B_{U(n)},Z)$ is a polynomial algebra on the universal chern classes $c_1,\ldots,c_n$ where $c_i \in H^{2i}(B_{U(n)},Z)$ and that $H^*(U(n),Z)$ is the exterior algebra with generators $x_1,\ldots,x_n$ where $x_i \in H^{2i-1}(U(n),Z)$ is the image under the map

$$\sigma : H^{2i}(B_{U(n)},Z) \longrightarrow H^{2i-1}(U(n),Z)$$

defined in 2.3.

4.2 Proposition

If $\xi = U(n) \longrightarrow E \longrightarrow SX$ is a principal $U(n)$ bundle over SX and triviallisations over the two cones differ by a map $\varphi : X \longrightarrow U(n)$ as in Prop.3.5, then for $i \geq 1$,

$$c_i(\xi) = \sigma^{-1}\varphi^*(x_i)$$

where $\sigma : H^{2i}(SX,Z) \longrightarrow H^{2i-1}(X,Z)$ is the suspension isomorphism.

Proof: By Prop.3.5, $\varphi : X \longrightarrow U(n)$ corresponds under the isomorphism of Theorem 1.5 to a map $\psi : SX \longrightarrow B_{U(n)}$ which classifies ξ. The proof then follows from the diagram

$$\begin{tikzcd}
H^{2i}(B_{U(n)}) \arrow[r, "\sigma"] \arrow[d, "\nu^*"'] & H^{2i-1}(U(n)) \arrow[d, "\varphi^*"] \\
H^{2i}(SX) \arrow[r, "\sigma"] & H^{2i-1}(X)
\end{tikzcd}$$

which commutes by Prop.2.4.

////.

5. Collapsing bundles.

5.1 Since the inclusion $U(n) \longrightarrow GL_n(C)$ is a deformation retraction it induces an isomorphism

$$H^*(GL_n(C)) \longrightarrow H^*(U(n))$$

so there is a unique element x_i' in $H^{2i-1}(GL_n(C))$ which maps onto the generator x_i of $H^{2i-1}(U(n))$. The main purpose of this section is to prove the following theorem:

5.2 Theorem

Suppose (X,A) has the homotopy extension property, and let E be a complex vector bundle over X which is trivial over A. Let

$$\alpha : E|A \longrightarrow A \times C^n$$

$$\beta : E|A \longrightarrow A \times C^n$$

be two trivialisations, and define $f: A \longrightarrow GL_n(C)$ by the equation

$$\alpha^{-1}(a,v) = \beta^{-1}(a,f(a).v)$$

Then for $i \geq 1$,

$$c_i(E/\alpha) - c_i(E/\beta) = \delta f^* x_i'$$

where $\delta : H^{2i-1}(A) \longrightarrow H^{2i}(X/A)$ is the coboundary, and E/α denotes the bundle over X/A obtained by the 'collapsing construction'.

First we observe that we may assume without loss of generality that $f(A) \subseteq U(n)$. For let $r: GL_n(C) \longrightarrow U(n)$ be a deformation, and let $g = rf: A \longrightarrow U(n)$. Let $h: A \times C^n \longrightarrow A \times C^n$ be the isomorphism

$$h(a,v) = (a, g(a)f(a)^{-1}.v)$$

Then $$(a,g(a).v) = h(a,f(a).v)$$
$$= h\beta\alpha^{-1}(a,v)$$

Put $\beta_1 = h\beta$. Then this says
$$\alpha^{-1}(a,v) = \beta_1^{-1}(a,g(a).v)$$

and now $g:A \longrightarrow U(n)$. But since $f \simeq g:A \longrightarrow GL_n(C)$, it follows that h is homotopic to the identity as vector bundle isomorphisms, and hence β is homotopic to β_1 as vector bundle isomorphisms. This implies that E/β is isomorphic to E/β_1 , and so for the proof of the theorem, we may assume that f = g and $\beta = \beta_1$.

5.4 For the rest of this section we assume that (X,A) has the homotopy extension property, that E is a complex n-dimensional vector bundle over X with trivialisations α, β over A (as in Th.5.2) and that $f:A \longrightarrow U(n)$ is such that
$$\alpha^{-1}(a,v) = \beta^{-1}(a,f(a).v)$$

5.5 Notation

If $Y = Y_1 \cup Y_2$ and F_i is a vector bundle over Y_i and
$$k:F_1 | Y_1 \cap Y_2 \longrightarrow F_2 | Y_1 \cap Y_2$$
is an isomorphism, then the bundle obtained over Y by the clutching construction is denoted by
$$F_1 \cup_k F_2.$$

We denote by $\underline{n}$ the trivial bundle of dimension n.

5.6 Proposition

Let $p:X \cup CA \longrightarrow X/A$ be the collapsing map. Then
$$p^*(E/\alpha) = E \cup_\alpha \underline{n}$$
where $\underline{n}$ is the bundle over CA.

Proof: Let $q:A \times C^n \longrightarrow C^n$ be projection. For $e,e' \in E$, we define an equivalence relation $e \sim e'$ if $e = e'$ or $q\alpha(e) = q\alpha(e')$. The identification space is (by definition) E/α . Let $h:E \longrightarrow E/\alpha$ be the identification map.

Define $k: CA \times C^n = \underline{n} \longrightarrow E/\alpha$ by the equation

$$k([a,t],v) = h\alpha^{-1}(a,v).$$

Then h and k define a vector bundle map from $E \amalg (CA \times C^n)$ to E/α covering the map $X \amalg CA \longrightarrow X/A$, which is an isomorphism on each fibre.

$$\begin{tikzcd}
& E \amalg (CA \times C^n) \arrow[rr] \arrow[dl] \arrow[dd, dashed] & & E/\alpha \arrow[dd, dashed] \\
E \cup_\alpha (CA \times C^n) \arrow[urrr] \arrow[dd] & & & \\
& X \amalg CA \arrow[rr] \arrow[dl] & & X/A \\
X \cup CA \arrow[urrr, "p"'] & & &
\end{tikzcd}$$

Moreover this map respects the clutching map α and so induces a vector bundle map from $E \cup_\alpha \underline{n}$ to E/α covering p, which is an isomorphism on each fibre.

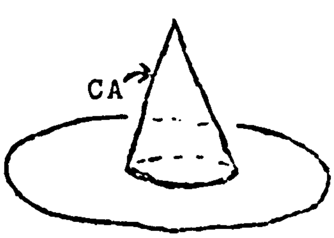
Hence $p^*(E/\alpha) = E \cup_\alpha \underline{n}$.

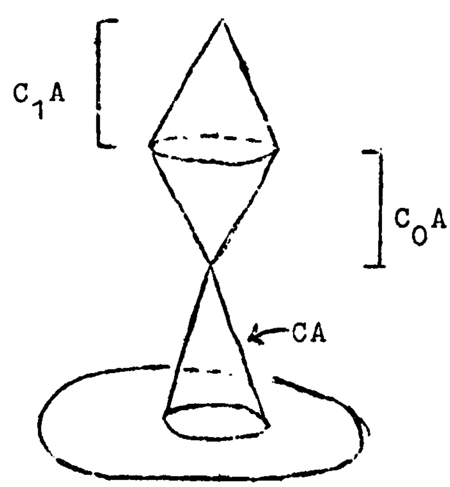
////.

<u>5.7</u> Let $d: X \cup CA \longrightarrow (X \cup CA) \vee SA$ be the pinching map, i.e.

$$\begin{aligned}
d(x) &= x && \text{for } x \in X \\
d[a,t] &= [a,2t] \in CA && \text{for } 0 \leq t \leq \tfrac{1}{2} \\
d[a,t] &= [a,2t-1] \in SA && \text{for } \tfrac{1}{2} \leq t \leq 1.
\end{aligned}$$

Let C_0A and C_1A be the lower and upper half cones of SA.





5.8 Lemma

Let F be a complex vector bundle over X, and let $\eta : F|A \longrightarrow A \times C^n$ be a trivialisation. Let $\zeta : A \times C^n \longrightarrow A \times C^n$ be a bundle isomorphism and let G be the bundle on $(X \cup CA) \vee SA$ which is

$$F \cup_{\eta} (CA \times C^n) \quad \text{on } X \cup CA$$

$$(C_0 A \times C^n) \cup_{\zeta} (C_1 A \times C^n) \quad \text{on } SA$$

Then $d^*G \cong F \cup_{\zeta\eta} (CA \times C^n)$

Proof: We define a vector bundle map from $F \cup_{\zeta\eta} (CA \times C^n)$ to G which covers d and is an isomorphism on each fibre. This map is defined in four parts:

(i) $F \longrightarrow F$ over X is the identity.

(ii) $A \times [0,\tfrac{1}{2}] \times C^n \longrightarrow CA \times C^n$ is

$$(a,t,v) \longmapsto (d[a,t], \zeta_a^{-1}(v))$$

(iii) $A \times [1/2,3/4] \times C^n \longrightarrow C_0 A \times C^n$ is

$$(a,t,v) \longmapsto (d[a,t], \zeta_a^{-1}(v))$$

(iv) $\dfrac{A \times [3/4,1]}{A \times \{1\}} \longrightarrow C_1 A \times C^n$ is

$$(a,t,v) \longmapsto (d[a,t], v)$$

where $\zeta_a : C^n \longrightarrow C^n$ is defined by the equation

$$(a, \zeta_a(v)) = \zeta(a,v)$$

These four maps combine with the various identifications to give the map as required.

////

We need one more lemma before proving the theorem (5.1)

5.9 Lemma

Let $i : X \cup CA \longrightarrow (X \cup CA) \vee SA$ and $j : SA \longrightarrow (X \cup CA) \vee SA$ be the inclusions, and let $q : X \cup CA \longrightarrow SA$ be the map collapsing X. Then for $r \geq 2$, the following diagram commutes:

$$\begin{tikzcd} H^r((X\cup CA)\vee SA) \arrow[r, dashed, "d^*"] \arrow[d, dashed, "\binom{i^*}{j^*}"'] & H^r(X\cup CA) \\ H^r(X\cup CA)\oplus H^r(SA) \arrow[r, dashed, "1\oplus q^*"'] & H^r(X\cup CA)\oplus H^r(X\cup CA) \arrow[u, "\text{addition}"'] \end{tikzcd}$$

Proof: Since $r \geq 2$, the map $\binom{i^*}{j^*}$ is an isomprphism. Let

$$r:(X\cup CA)\vee SA \longrightarrow X\cup CA$$
$$s:(X\cup CA)\vee SA \longrightarrow SA$$

be the maps collapsing SA and $X\cup CA$ respectively. Then

$$(r^*,s^*):H^r(X\cup CA)\oplus H^r(SA) \longrightarrow H^r((X\cup CA)\vee SA)$$

is the inverse map, and so it suffices to show that the following diagram commutes:

$$\begin{tikzcd} H^r((X\cup CA)\vee SA) \arrow[r, "d^*"] & H^r(X\cup CA) \\ H^r(X\quad CA)\oplus H^r(SA) \arrow[u, "(r^*,s^*)"] \arrow[r, "1\oplus q^*"'] & H^r(X\cup CA)\oplus H^r(X\cup CA) \arrow[u, "\text{addition}"'] \end{tikzcd}$$

Take $x\oplus y \in H^r(X\cup CA)\oplus H^r(SA)$. Then

$\lrcorner$ takes $x\oplus y$ to $x + q^*y$

$\ulcorner$ takes $x\oplus y$ to $d^*r^*x + d^*r^*y$.

But $rd \simeq 1:X\cup CA \longrightarrow X\cup CA$ and $sd \simeq q:X\cup CA \longrightarrow SA$, so

$$d^*r^*x + d^*s^*y = x + q^*y$$

and the diagram commutes.

////.

5.10 Proof of Theorem 5.2

Let M be the bundle on $(X\cup CA)\vee SA$ which is

5.10.1 $\quad E\cup_\alpha(CA\times C^n)$ on $X\cup CA$

$$F = (C_0A\times C^n)\cup_f(C_1A\times C^n) \text{ on } SA$$

Then by Prop.5.8,

$$d^*M \cong E\cup_\beta(CA\times C^n) \qquad \text{since } \beta\alpha^{-1} = f.$$

Let $g:X\cup CA \longrightarrow B_{U(n)}$ classify $E\cup_\alpha(CA\times C^n)$, and let

$\hat{f}:SA \longrightarrow B_{U(n)}$ classify the bundle F over SA.

Then by Prop.3.5, the class of $\hat{f}$ corresponds to the class of f under the isomorphism $[SA, B_{U(n)}] \cong [A, U(n)]$.

Then M is classified by the map

$$(g\vee\hat{f})d : (X\cup CA)\vee SA \longrightarrow B_{U(n)}$$

and so

$$c_i(E\cup_\beta(CA\times C^n)) - c_i(E\cup_\alpha(CA\times C^n))$$
$$= ((g\vee\hat{f})d)^* c_i - g^* c_i.$$

Now by Lemma 5.9

$$((g\vee\hat{f})d)^* c_i = d^*(g\vee\hat{f})^* c_i$$
$$= g^* c_i + q^*\hat{f}^* c_i .$$

$$\begin{tikzcd}
H^{2i}(B_{U(n)}) \arrow[r, "(g\vee\hat{f})^*"] \arrow[dr, "\binom{g^*}{\hat{f}^*}"'] & H^{2i}((X\cup CA)\vee SA) \arrow[r, "d^*"] \arrow[d, "\binom{i^*}{j^*}"] & H^{2i}(X\cup CA) \\
 & H^{2i}(X\cup CA)\oplus H^{2i}(SA) \arrow[r, "1\oplus q^*"'] & H^{2i}(X\cup CA)\oplus H^{2i}(X\cup CA) \arrow[u, "\text{add}"']
\end{tikzcd}$$

Thus

$$c_i(E\cup_\beta(CA\times C^n)) - c_i(E\cup_\alpha(CA\times C^n)) = q^*\hat{f}^* c_i .$$

By Prop.5.6 this equation can be written

$$c_i(p^*(E/\beta)) - c_i(p^*(E/\alpha)) = q^*\hat{f}^* c_i$$

or $\quad p^*(\, c_i(E/\beta) - c_i(E/\alpha)\,) = q^*\hat{f}^* c_i.$

The proof now follows from Prop.2.4 and the following commutative diagram.

$$\begin{tikzcd}
H^{2i}(B_{U(n)}) \arrow[r, "\sigma"] \arrow[d, "\hat{f}^*"] & H^{2i-1}(U(n)) \arrow[d, "f^*"] \\
H^{2i}(SA) \arrow[r, "\sigma", "\cong"'] \arrow[d, "q^*"'] & H^{2i-1}(A) \arrow[d, "\delta"] \\
H^{2i}(X\cup CA) & H^{2i}(X/A) \arrow[l, "\cong"', "p^*"]
\end{tikzcd}$$

////.

6. Examples

6.1 Suppose (X,A) has the homotopy extension property, and $f:A \longrightarrow U(n)$ is some map. Then f induces a vector bundle say $d(f)$ over X/A as follows. Let $\underline{n}$ denote the trivial n dimensional bundle over X. Then f defines a trivialisation $\underline{n}|A \xrightarrow{\hat{f}} A \times C^n$ by

$$(a,v) \longmapsto (a,f(a).v)$$

and the collapsing construction then defines the bundle $\underline{n}/\hat{f} = d(f)$. (Remark: This is a particular case of the construction of the 'difference bundle' as in (5).)

6.2 Proposition

If (X,A) *has the HEP and* $f:A \longrightarrow U(n)$, *then*

$$c_i(d(f)) = \delta f^*x_i \qquad i \geq 1$$

Proof: Let $1:\underline{n}|A \longrightarrow A \times C^n$ be the identity trivialisation. Then in the data of Theorem 5.2 take $\alpha = 1$, and $\beta = \hat{f}$. Then

$$c_i(d(f)) - c_i(d(1)) = \delta f^*x_i$$

but $d(1)$ is the trivial bundle, so $c_i(d(1)) = 0$ for $i \geq 1$.

////.

6.3 Proposition

Suppose (X,S^{2n-1}) *has the homotopy extension property and that*

(i) $H^{2n-1}(X) = 0 = H^{2n}(X)$

(ii) m is an integer such that the only $U(m)$ *bundle over* X *is the trivial one.*

Then from (i) $H^{2n}(X/S^{2n-1}) = Z$, *and the n-th Chern class of any* $U(m)$ *bundle over* X/S^{2n-1} *is a multiple of* $(n-1)!$

Proof: Any $U(m)$ bundle over X/S^{2n-1} is obtained by collapsing a $U(m)$ bundle on X. But by (ii) there is only one $U(m)$ bundle on X, so any $U(m)$ bundle on X/S^{2n-1} is obtained by the construction of 6.1, and so by Prop.6.2, the n-th Chern class is of the form

$$\delta f^* x_n \quad \text{for some map } f: A \longrightarrow U(m)$$

Since δ is an isomorphism by (i), it suffices to prove the following lemma.

6.4 Lemma

Suppose $f: S^{2n-1} \longrightarrow U(m)$ with $m \geq n$ induces $f^*: H^{2n-1}(U(m)) \longrightarrow H^{2n-1}(S^{2n-1})$

Then $f^* x_n$ is a multiple of $(n-1)!$

Proof: The inclusion $U(n) \longrightarrow U(m)$ induces isomorphisms

$$\pi_{2n-1}(U(n)) \longrightarrow \pi_{2n-1}(U(m))$$

$$\text{and} \quad H^{2n-1}(U(m)) \longrightarrow H^{2n-1}(U(n))$$

taking x_n to x_n (Atiyah and Todd (6)), hence we may assume without loss of generality that $m = n$. Consider the fibration

$$U(n-1) \xrightarrow{i} U(n) \xrightarrow{p} S^{2n-1}$$

It is well known (Atiyah and Todd (6)) that

$$p^*: H^{2n-1}(S^{2n-1}) \longrightarrow H^{2n-1}(U(n)) \text{ is an}$$

is an isomorphism, so it suffices to show that the map

$$pf: S^{2n-1} \longrightarrow S^{2n-1}$$

has degree a multiple of $(n-1)!$

From the exact sequence of the fibration we have

$$\pi_{2n-1}(U(n)) \xrightarrow{p_*} \pi_{2n-1}(S^{2n-1}) \xrightarrow{\partial} \pi_{2n-2}(U(n-1)) \xrightarrow{i_*} \pi_{2n-2}(U(n))$$

But

$$\pi_{2n-1}(S^{2n-1}) = Z$$

$$\pi_{2n-2}(U(n-1)) = Z_{(n-1)!}$$

$$\pi_{2n-2}(U(n)) = Z$$

and since $\mathrm{Hom}(Z_{(n-1)!}, Z) = 0$, i^* is zero, so ∂ is an epimorphism. Since $\partial[pf] = \partial p_*[f] = 0$, $pf \in \mathrm{Ker}\,\partial = (n-1)!Z$, whence the degree of pf is a multiple of $(n-1)!$

////.

Prop.6.3 has a simple generalisation. We say that an element a of an abelian group A is a multiple of r (where r is some integer) if there is an element $b \in A$ such that $a = rb$.

6.5 Proposition

Suppose that (X, S^{2n-1}) has the HEP, and m is an integer such that the only $U(m)$ bundle over X is the trivial one. Then the n-th Chern class of any $U(m)$ bundle over X/S^{2n-1} is a multiple of $(n-1)!$

Proof: As in Prop6.2, the n-th Chern class is of the form

$$\delta f^* x_n \quad \text{for some map } f : S^{2n-1} \longrightarrow U(m).$$

By Lemma 6.3, $f^* x_n$ is a multiple of $(n - 1)!$, so $\delta f^* x_n$ is a multiple of $(n - 1)!$

////.

6.6 An obvious example of Prop.6.5 is the pair (D^{2n}, S^{2n-1}) with m arbitrary. This gives us the well known result that the n-th Chern class of any vector bundle over S^{2n} is amultiple of $(n - 1)!$

Moreover, this proposition also implies that every such multipl occurs. For let $g : S^{2n-1} \longrightarrow S^{2n-1}$ be a map of degree $(n - 1)!$ Then from the exact sequence 6.3.1, $\partial(g) = 0$, so there is a map $f : S^{2n-1} \longrightarrow U(n)$ such that $pf \simeq g$. We no show that the bundle $d(f)$ over $D^{2n}/S^{2n-1} = S^{2n}$ has n-th Chern class equal to $(n-1)!$ times a generator. By Prop.6.2,

$$c_n(d(f)) = \delta f^* x_n$$

where $\delta : H^{2n-1}(S^{2n-1}) \longrightarrow H^{sn}(D^{2n}/S^{2n-1}) = H^{2n}(S^{2n})$

is now the suspension isomorphism. It suffices then to show that $f^* x_n$ is $(n-1)!$ times a generator of $H^{2n-1}(S^{2n-1})$ and this follows from the commutative diagram

$$\begin{tikzcd}
H^{2n-1}(U(n)) \arrow[rr, "f^*"] & & H^{2n-1}(S^{2n-1}) \\
& H^{2n-1}(S^{2n-1}) \arrow[ul, "p^*"] \arrow[ur, "g^*"'] &
\end{tikzcd}$$

since g^* is multiplication by $(n-1)!$. and p^* is an isomorphism.

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CHAPTER 4 : TRACE IN ADDITIVE CATEGORIES

Trace in additive categories

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Introduction

Given an endomorphism of a finite dimensional vector space we can define its trace by choosing a basis and taking the trace of the corresponding matrix. One then shows that this is independent of the choice of basis, and so one might expect that it was possible to define the trace in a natural way which does not involve making a choice. In fact it is well known that this is possible. If V is a finite dimensional vector space over the field F, then there is a natural isomorphism

$$\Theta : \mathrm{End}_F(V) \longrightarrow V \otimes_F V^*$$

If $e : V \otimes_F V^* \longrightarrow F$ denotes the evaluation map, then the trace of an endomorphism $v : V \longrightarrow V$ is $e\Theta(v)$. (In fact this construction will define the trace for endomorphisms of finitely generated projective modules over a commutative ring. See Bourbaki[1], §4, pp 105-113)

This trace, apparently defined for one vector space at a time then satisfies a compatibility condition: given a commutative diagram of finite dimensional vector spaces with exact rows

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, dotted, "e"'] & H \arrow[r] \arrow[d, "h"'] & G \arrow[r] \arrow[d, "g"'] & 0 \\
0 \arrow[r] & E \arrow[r] & H \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

then $\mathrm{trace}(e) + \mathrm{trace}(g) = \mathrm{trace}(h)$. There are two other important properties of trace. Firstly it is a homomorphism, and secondly if e, f are endomorphisms of the same vector space, then $\mathrm{trace}(ef)$ and $\mathrm{trace}(fe)$ are equal. Starting with these three conditions as axioms and a normalisation condition (that trace is to be the right thing on endomorphisms of F namely the trace of an endomorphism is the endomorphism) one can prove that the usual trace is the unique function satisfying the axioms.

This gives us a starting point for defining a trace in an additive category, and this paper shows that under certain finiteness conditions on the category there is a unique trace obeying some normalisation condition. Several examples are given in section 7. The paper then considers a topological version of the theorem with an application to complex vector bundles which shows that a continuous trace corresponds to a regular Borel measure.

A version of this paper has appeared ~~is to appear~~ in the Proceedings of the Cambridge Philosophical Society.

1. Additive categories and traces

A category $\underline{N}$ is additive if

(i) $\underline{N}$ has finite sums and products

(ii) $\underline{N}$ has a zero object

(iii) for each $A,B \in \mathrm{Ob}\,\underline{N}$ the set $\underline{N}(A,B)$ of morphisms from A to B is an abelian group such that composition is bilinear.

A sequence

$$0 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ p\ } C \longrightarrow 0$$

is said to be exact if $i = \ker p$ and $p = \operatorname{coker} i$. We write End (A) for $\underline{N}(A,A)$ or we may write $\mathrm{End}_{\underline{N}}(A)$ if it is not clear which category we are taking the morphisms in.

1.1 Definition

A trace on $\underline{N}$ with values in an abelian group G is a collection of (abelian group) homomorphisms

$$t = \{t_A : \mathrm{End}\ (A) \longrightarrow G\}$$

one for each $A \in \mathrm{Ob}\,\underline{N}$, satisfying the following two conditions:

(i) Exactness: Given a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "a"'] & B \arrow[r] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"'] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

then $\quad t_A(a) + t_C(c) = t_B(b).$

(ii) Commutativity: If $f,g \in \mathrm{End}\ (A)$ then

$$t_A(fg) = t_A(gf).$$

The collection of zero homomorphisms constitute a trace. Given two traces, we can define their sum componentwise: if $t = \{t_A : \mathrm{End}\ (A) \longrightarrow G\}$ and $s = \{s_A : \mathrm{End}\ (A) \longrightarrow G\}$ then

$$t + s = \{t_A + s_A : \mathrm{End}\ (A) \longrightarrow G\}$$

The set of traces on $\underline{N}$ with values in G thus form an abelian group denoted by $\mathrm{Tr}(\underline{N},G)$. If $F:\underline{N} \longrightarrow \underline{M}$ is an exact functor between additive categories (and hence an additive functor) then F induces a homomorphism $F^*:Tr(\underline{M},G) \longrightarrow Tr(\underline{N},G)$ by composition:

$$F^* \left\{ t_A:\mathrm{End}_{\underline{M}}(A) \longrightarrow G \right\} = \left\{ t_{F(B)}F:\mathrm{End}_{\underline{N}}(B) \longrightarrow \mathrm{End}_{\underline{M}}(F(B)) \longrightarrow G \right\}$$

<u>1.2 Definition:</u> For $A \in \mathrm{Ob}\ \underline{N}$, let $T(A) = \mathrm{End}\ (A)/\ S(A)$ where $S(A)$ denotes the subgroup of End (A) generated by elements of the form $fg - gf$.

For $A \in \mathrm{Ob}\ \underline{N}$, let $\eta_A:Tr(\underline{N},G) \longrightarrow \mathrm{Hom}\ (T(A),G)$ by $\eta_A(t) = t_A$, where by abuse of notation t_A also denotes the homomorphism from $T(A)$ to G induced by t_A. (By 1.1 (ii) $t_A:\mathrm{End}\ (A) \longrightarrow G$ is zero on $S(A)$). Clearly η_A is a homomorphism.

<u>1.3 Definition:</u> If $A,K \in \mathrm{Ob}\ \underline{N}$ then A is finitely generated over K if there exists an epimorphism $K^n \longrightarrow A$ for some $n \geq 0$. A is K-free if A is isomorphic with K^n for some $n \geq 0$.

<u>1.4 Definition</u>

Let $\underline{N}$ be an additive category, and $K \in \mathrm{Ob}\ \underline{N}$ a projective object. then $(\underline{N},K)$ is a category

- of type (a) if every object is K-free
- of type (b) if every object is a direct summand in a K-free object
- of type (c) if $\underline{N}$ is abelian, every object in $\underline{N}$ is finitely generated over K and has finite homological dimension.

The main theorem in this paper is the following:

<u>1.5 Theorem</u>

<u>Let $\underline{N}$ be an additive category and $K \in \mathrm{Ob}\ \underline{N}$ a projective object. then if $(\underline{N},K)$ is a category of type (a), (b) or (c) then</u>

$$\eta_K:Tr(\underline{N},G) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(T(K),G)$$

<u>is an isomorphism.</u>

For categories of type (a) or (b), that η_K is surjective follows from the definition and construction of a trace function for square matrices over arbitrary rings by Stallings (3). Following this we call the trace in $Tr(\underline{N},T(K))$ corresponding under the above isomorphism to the identity map on $T(K)$ the K-universal trace.

The theorem is proved for categories of type (a) in section 2, for categories of type (b) in section 3, and categories of type (c) in section 4. The result is extended to linear categories in section 5 and topological categories in section 7. Examples are given in sections 6 and 8.

2. The theorem for categories of type (a).

Let $(\underline{A},K)$ be a category of type (a). We first prove that η_K is surjective. Let $R = End(K)$, and let $\underline{N}$ be the category of finitely generated free (right) R-modules. The functor

$$\underline{A}(K,-):\underline{A} \longrightarrow \underline{N}$$

is exact so induces a homomorphism from $Tr(\underline{N},G)$ to $Tr(\underline{A},G)$ such that the following diagram commutes:

$$\begin{tikzcd} Tr(\underline{N},G) \arrow[rr] \arrow[dr, "\eta_R"'] & & Tr(\underline{A},G) \arrow[dl, dashed, "\eta_K"] \\ & Hom_{\mathbb{Z}}(T(K),G) & \end{tikzcd}$$

It follows from Stallings ((3), 1.6) that η_R is surjective. For suppose that $t:T(K) \longrightarrow G$ is a homomorphism. Since $T(K)$ is the additive group of R modulo the subgroup generated by elements of the form $(fg - gf)$ we can think of t as being defined on R and zero on this subgroup. Now suppose F is a finitely generated free R-module, and $\phi:F \longrightarrow F$ is an R-homomorphism. Choose a basis $\{f_1,\ldots,f_n\}$ for F with respect to which we can represent ϕ by a matrix ϕ_{ij} where

$$\phi(f_i) = \sum_j f_j\phi_{ij}$$

Now let $t_F(\phi) = \sum_i t(\phi_{ii})$. This is independent of the choice of basis and defines a homomorphism $t_F:\mathrm{End}_R(F) \longrightarrow G$. The collection of these homomorphisms constitutes a trace, which shows that η_R is surjective and hence η_K is surjective.

To prove that η_K is injective, suppose that for some $t \in \mathrm{Tr}(\underline{A},G)$ we have $\eta_K(t) = t_K = 0$. By considering exact sequences of the form

$$0 \longrightarrow K \longrightarrow K^n \longrightarrow K^{n-1} \longrightarrow 0$$

it easily follows inductively that $t_{K^n} = 0$, and since every object in $\underline{A}$ is K-free, $t_A = 0$ for all $A \in \mathrm{Ob}\ \underline{A}$. This completes the proof of the theorem in case (a).

3. The theorem for categories of type (b).

Let $(\underline{B},K)$ be a category of type (b). Let $\underline{M}$ be the category of finitely generated projective (right) R-modules, where again R is End (K). The functor

$$\underline{B}(K,-):\underline{B} \longrightarrow \underline{M}$$

is exact and so by similar argument to section 2 and using ((3),1.7) we see that η_K is surjective. For suppose P is a finitely generated projective R-module, and $\emptyset:P \longrightarrow P$ is an R-homomorphism, and $t:T(K) \longrightarrow G$. Choose a $Q \in \mathrm{Ob}\ \underline{M}$ such that $P \oplus Q$ is free, and define $t_P(\emptyset) = t_{P\oplus Q}(\emptyset \oplus 0)$, where $t_{P\oplus Q}$ denotes the trace on the free R-module $P \oplus Q$ constructed as in section 2. This does not depend on the choice of Q and defines a homomorphism $t_P:\mathrm{End}\ (P) \longrightarrow G$, and the collection of these homomorphisms constitutes a trace which shows that η_R is surjective, and hence η_K is surjective.

To prove that η_K is injective, let $\underline{A}$ be the full subcategory of $\underline{B}$ of K-free objects. Then $(\underline{A},K)$ is a category of type (a). Suppose that $t = \{ t_B:\mathrm{End}_{\underline{B}}(B) \longrightarrow G\} \in \mathrm{Tr}(\underline{B},G)$ and that $\eta_K(t) = 0$. Then t restricted to the subcategory $\underline{A}$ is a trace, and so by the theorem in case (a) we know that $t_A = 0$ for all $A \in \mathrm{Ob}\ \underline{A}$. Suppose that

$B \in \mathrm{Ob}\,\underline{B}$. Then there exists a $Q \in \mathrm{Ob}\,\underline{B}$ such that $B \oplus Q \in \mathrm{Ob}\,\underline{A}$. Let $i : B \longrightarrow B \oplus Q$ and $p : B \oplus Q \longrightarrow Q$ be the injection and projection maps. If $b \in \mathrm{End}_{\underline{B}}(B)$ then the following diagram commutes and has exact rows:

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "i"] \arrow[d, "b"'] & B\oplus Q \arrow[r, "p"] \arrow[d, "b\oplus 0"'] & Q \arrow[r] \arrow[d, "0"'] & 0 \\
0 \arrow[r] & B \arrow[r, "i"'] & B\oplus Q \arrow[r, "p"'] & Q \arrow[r] & 0
\end{tikzcd}$$

Hence $t_B(b) + t_Q(0) = t_{B\oplus Q}(b\oplus 0) = 0$ since $B\oplus Q \in \mathrm{Ob}\,\underline{A}$, and so $t_B(b) = 0$ for all $b \in \mathrm{End}_{\underline{B}}(B)$, whence $t_B = 0$ for all $B \in \mathrm{Ob}\,\underline{B}$.

This completes the proof of the theorem for categories of type (b).

4. The theorem for categories of type (c).

Let $(\underline{C},K)$ be a category of type (c), and let $\underline{B}$ be the full subcategory of projective objects. Then $(\underline{B},K)$ is of type (b). If $B \in \mathrm{Ob}\,\underline{B}$ then $\mathrm{End}_{\underline{B}}(B) = \mathrm{End}_{\underline{C}}(B)$ since $\underline{B}$ is a full subcategory. In particular $T(K)$ is well defined.

We first show that $\eta_K : \mathrm{Tr}(\underline{C},G) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(T(K),G)$ is surjective. Suppose $t_K \in \mathrm{Hom}_{\mathbb{Z}}(T(K),G)$. Then by the theorem applied to $(\underline{B},K)$ there is a unique trace $t \in \mathrm{Tr}(\underline{B},G)$ such that $\eta_K(t) = t_K$. Suppose $M \in \mathrm{Ob}\,\underline{C}$. Then there is a finite projective resolution of M, say

4.1 $$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

which we write $\varepsilon : \underset{\sim}{P} \longrightarrow M$. If $f \in \mathrm{End}\,(M)$, then we may lift f to a chain map

4.2 $$\begin{tikzcd}
0 \arrow[r] & P_n \arrow[r] \arrow[d, "f_n"'] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "f_0"'] & M \arrow[r] \arrow[d, "f"'] & 0 \\
0 \arrow[r] & P_n \arrow[r] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

which we write $\underset{\sim}{f} : \underset{\sim}{P} \longrightarrow \underset{\sim}{P}$.

4.3 Define $\quad t_M(f) \;=\; \sum_i (-1)^i\, t_{P_i}(f_i).$

In order to show that t_M is well defined, we shall need the following lemma.

4.4 Lemma

If $P, Q \in \operatorname{Ob} \underline{B}$ and $f: P \longrightarrow Q$ and $g: Q \longrightarrow P$ then $t_P(gf) = t_Q(fg)$.

Proof: Let $f': P \oplus Q \longrightarrow P \oplus Q$ be the map

$$\begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix}$$

and $g': P \oplus Q \longrightarrow P \oplus Q$ be the map

$$\begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix}$$

Then $\quad t_{P \oplus Q}(g'f') \;=\; t_{P \oplus Q}(f'g').$ But $g'f': P \oplus Q \longrightarrow P \oplus Q$ is the map

$$\begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix} = \begin{pmatrix} gf & 0 \\ 0 & 0 \end{pmatrix}$$

and so $t_{P \oplus Q}(g'f') = t_P(gf)$ by exactness (1.1.(i)). Similarly $t_{P \oplus Q}(f'g') = t_Q(fg)$ and the result follows. ////.

4.5 Lemma

(i) $t_M(f)$ *is independent of the choice of lifting.*

(ii) $t_M(f)$ *is independent of the choice of resolution.*

Proof: (i). If $\underset{\sim}{f}: \underset{\sim}{P} \longrightarrow \underset{\sim}{P}$ and $\underset{\sim}{f}': \underset{\sim}{P} \longrightarrow \underset{\sim}{P}$ are liftings of $f: M \longrightarrow M$ they are chain homotopic, so there exists a family of maps

$$s_i: P_i \longrightarrow P_{i+1}$$

such that

4.5.1 $$f_i - f_i' \;=\; d_{i+1}s_i \;+\; s_{i-1}d_i$$

$$\begin{tikzcd}
 & P_i \arrow[r, "d_i"] \arrow[dl, dashed, "s_i"'] & P_{i-1} \arrow[dl, dashed, "s_{i-1}"] \\
P_{i+1} \arrow[r, "d_{i+1}"'] & P_i \arrow[r, "d_i"'] & P_{i-1}
\end{tikzcd}$$

So $\quad t_{P_i}(f_i) - t_{P_i}(f'_i) \;=\; t_{P_i}(d_{i+1}s_i) + t_{P_i}(s_{i-1}d_i)$ and hence

$$\sum_i (-1)^i t_{P_i}(f_i) - \sum_i (-1)^i t_{P_i}(f'_i)$$
$$= \sum_i (-1)^i t_{P_i}(d_{i+1}s_i) - \sum_i (-1)^i t_{P_i}(s_{i-1}d_i)$$

But the right hand side is zero, since by Lemma 4.4

$$t_{P_i}(s_{i-1}d_i) = t_{P_{i-1}}(d_i s_{i-1}).$$

This proves 4.5.(i).

Proof (ii). Suppose $\varepsilon : \underset{\sim}{P} \longrightarrow M$ and $\varepsilon' : \underset{\sim}{P}' \longrightarrow M$ are resolutions of finite length by objects in $\underline{B}$. Applying the Comparison Lemma,

lift $f : M \longrightarrow M$ to a chain map $\underset{\sim}{f} : \underset{\sim}{P} \longrightarrow \underset{\sim}{P}$

lift $1 : M \longrightarrow M$ to a chain map $\underset{\sim}{x} : \underset{\sim}{P} \longrightarrow \underset{\sim}{P}'$

lift $1 : M \longrightarrow M$ to a chain map $\underset{\sim}{y} : \underset{\sim}{P}' \longrightarrow \underset{\sim}{P}$.

Then $\underset{\sim}{f}\underset{\sim}{y}\underset{\sim}{x} : \underset{\sim}{P} \longrightarrow \underset{\sim}{P}$ and $\underset{\sim}{x}\underset{\sim}{f}\underset{\sim}{y} : \underset{\sim}{P}' \longrightarrow \underset{\sim}{P}'$ each lift $f : M \longrightarrow M$.

By Lemma 4.4 applied to the maps $x_i : P_i \longrightarrow P'_i$ and $f_i y_i : P'_i \longrightarrow P_i$ we see that

$$t_{P_i}(f_i y_i x_i) = t_{P'_i}(x_i f_i y_i)$$

so that

$$\sum_i (-1)^i t_{P_i}(f_i y_i x_i) \;=\; \sum_i (-1)^i t_{P'_i}(x_i f_i y_i).$$

But the left hand side defines $t_M(f)$ via the resolution $\underset{\sim}{P}$, and the right hand side via $\underset{\sim}{P}'$, using 4.5.(i), so that $t_M(f)$ is well defined.

////.

Clearly $t_M : \text{End}\,(M) \longrightarrow G$ is a homomorphism such that if $f, g \in \text{End}\,(M)$ then $t_M(gf) = t_M(fg)$, since t_M is defined to be a sum of homomorphisms each of which satisfies this commutatativity condition. So in order to show that the collection of these homomorphisms constitutes a trace, we have to prove exactness (1.1.(i)).

Suppose that

4.6.1
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, dashed, "a"'] & B \arrow[r] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"'] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram in the category $\underline{C}$, with exact rows. It is well known that given projective resolutions $\varepsilon_A : \underset{\sim}{P} \longrightarrow A$ and $\varepsilon_C : \underset{\sim}{R} \longrightarrow C$ of finite length, there is a projective resolution $\varepsilon_B : \underset{\sim}{Q} \longrightarrow B$ and a commutative diagram

4.6.2
$$\begin{tikzcd} \underset{\sim}{0} \arrow[r] & \underset{\sim}{P} \arrow[r, "\underset{\sim}{i}"] \arrow[d, dashed, "\varepsilon_A"'] & \underset{\sim}{Q} \arrow[r, "\underset{\sim}{p}"] \arrow[d, dashed, "\varepsilon_B"'] & \underset{\sim}{R} \arrow[r] \arrow[d, dashed, "\varepsilon_C"'] & \underset{\sim}{0} \\ 0 \arrow[r] & A \arrow[r, "i"'] & B \arrow[r, "p"'] & C \arrow[r] & 0 \end{tikzcd}$$

where $Q_j = P_j \oplus R_j$ and the maps $i_j : P_j \longrightarrow Q_j$ and $p_j : Q_j \longrightarrow R_j$ are the injection and projection maps.

Lift $a : A \longrightarrow A$ to a map $\underset{\sim}{a} : \underset{\sim}{P} \longrightarrow \underset{\sim}{P}$ and $c : C \longrightarrow C$ to a map $\underset{\sim}{c} : \underset{\sim}{R} \longrightarrow \underset{\sim}{R}$. Let the components of $\varepsilon_B : Q_0 = P_0 \oplus R_0 \longrightarrow B$ be $\alpha : P_0 \longrightarrow B$ and $\beta : R_0 \longrightarrow B$. Then

$$p(b\beta - \beta c_0) = cp\beta - p\beta c_0 = 0$$

so that $b\beta - \beta c_0 = i\mu_0$ for some map $\mu_0 : R_0 \longrightarrow A$. Since R_0 is projective and ε_A is an epimorphism, $\mu_0 = \varepsilon_A \theta_0$ for some map $\theta_0 : R_0 \longrightarrow P_0$. Let $b_0 : Q_0 \longrightarrow Q_0$ be the map

$$\begin{pmatrix} a_0 & \theta_0 \\ 0 & c_0 \end{pmatrix} : P_0 \oplus R_0 \longrightarrow P_0 \oplus R_0$$

Then $\varepsilon_B b_0 = b\,\varepsilon_B$. Proceeding by the usual inductive process we obtain a map $\underset{\sim}{b} : \underset{\sim}{Q} \longrightarrow \underset{\sim}{Q}$ which lifts $b : B \longrightarrow B$ and such that the following diagram of chain complexes commutes:

4.6.3
$$\begin{tikzcd} \underset{\sim}{0} \arrow[r] & \underset{\sim}{P} \arrow[r, "\underset{\sim}{i}"] \arrow[d, "\underset{\sim}{a}"'] & \underset{\sim}{Q} \arrow[r, dashed, "\underset{\sim}{p}"] \arrow[d, dashed, "\underset{\sim}{b}"'] & \underset{\sim}{R} \arrow[r] \arrow[d, "\underset{\sim}{c}"'] & \underset{\sim}{0} \\ \underset{\sim}{0} \arrow[r] & \underset{\sim}{P} \arrow[r, "\underset{\sim}{i}"'] & \underset{\sim}{Q} \arrow[r, "\underset{\sim}{p}"'] & \underset{\sim}{R} \arrow[r] & \underset{\sim}{0} \end{tikzcd}$$

It follows that $t_A(a) + t_C(c) = t_B(b)$ since for all $j \geqslant 0$ we have

$$t_{P_j}(a_j) + t_{R_j}(c_j) = t_{Q_j}(b_j).$$

Hence $t = \{ t_M : \mathrm{End}\,(M) \longrightarrow G \}$ is a trace, and $\eta_K(t) = t_K$, thus proving that η_K is surjective in case (c).

To show that η_K is injective, suppose that t is a trace on $\underline{C}$ such that $\eta_K(t) = t_K = 0$. Then by applying the theorem (1.5 case (b)) to the full subcategory $\underline{B}$, we know that $t_B = 0$ for all $B \in \mathrm{Ob}\,\underline{B}$.

Take any $M \in \mathrm{Ob}\,\underline{C}$. Then there is a finite projective resolution $\xi : \underset{\sim}{P} \longrightarrow M$ say of length n. Let

$$K_0 = \mathrm{Ker}(\, P_0 \longrightarrow M)$$

$$K_j = \mathrm{Ker}(\, P_j \longrightarrow P_{j-1}) \text{ for } j = 1,\ldots,n-2.$$

Suppose $f \in \mathrm{End}\,M$, then lift f to a chain map $\underset{\sim}{f} : \underset{\sim}{P} \longrightarrow \underset{\sim}{P}$, and let $g_j : K_j \longrightarrow K_j$ $(j = 0,\ldots,n-2)$ be the maps induced by $\underset{\sim}{f}$. We then have n commutative diagrams with exact rows:

4.7.1
$$\begin{tikzcd}
0 \arrow[r] & K_0 \arrow[r] \arrow[d, "g_0"'] & P_0 \arrow[r] \arrow[d, "f_0"'] & M \arrow[r] \arrow[d, "f"'] & 0 \\
0 \arrow[r] & K_0 \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

4.7.j
$$\begin{tikzcd}
0 \arrow[r] & K_{j-1} \arrow[r] \arrow[d, "g_{j-1}"'] & P_{j-1} \arrow[r] \arrow[d, "f_{j-1}"'] & K_{j-2} \arrow[r] \arrow[d, "g_{j-2}"'] & 0 \\
0 \arrow[r] & K_{j-1} \arrow[r] & P_{j-1} \arrow[r] & K_{j-2} \arrow[r] & 0
\end{tikzcd}$$ for $j = 2,\ldots,n-1$

4.7.n
$$\begin{tikzcd}
0 \arrow[r] & P_n \arrow[r] \arrow[d, "f_n"'] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & K_{n-2} \arrow[r] \arrow[d, "g_{n-2}"'] & 0 \\
0 \arrow[r] & P_n \arrow[r] & P_{n-1} \arrow[r] & K_{n-2} \arrow[r] & 0
\end{tikzcd}$$

From these we get n equations:

4.8.1 $$t_{K_0}(g_0) + t_M(f) = t_{P_0}(f_0)$$

4.8.j $$t_{K_{j-1}}(g_{j-1}) + t_{K_{j-2}}(g_{j-2}) = t_{P_{j-1}}(f_{j-1}) \quad \text{for } j = 2,\ldots,n-1$$

4.8.n $$t_{P_n}(f_n) + t_{K_{n-2}}(g_{n-2}) = t_{P_{n-1}}(f_{n-1})$$

Simce $t_{P_j} = 0$ for $j = 0,\ldots,n$ ($P_j \in$ Ob $\underline{B}$), we deduce that $t_M(f) = 0$ i.e. that $t_M = 0$ for all $M \in$ Ob $\underline{C}$, whence $1|_K$ is injective which completes the proof of Theorem 1.5 in case (c).

4.9 Remarks

4.9.1 In the proof of Theorem 1.5 we make no particular use of the group G in which the trace takes its values. We state an equivalent version of the theorem which makes the rôle of G clear.

Let $\underline{Ab}$ be the category of abelian groups, and $\underline{N}$ an additive category. Then $Tr(\underline{N},-):\underline{Ab} \longrightarrow \underline{Ab}$ is a functor.

Theorem 1.5'

Let $\underline{N}$ be an additive category and suppose there is a projective object $K \in$ Ob N such that $(\underline{N},K)$ is of type (a),(b) or (c). Then $Tr(\underline{N},-):Ab \longrightarrow Ab$ is representable, and $T(K)$ is a representing object. ////.

Since from Yoneda's Lemma we know that any two representing objects are isomorphic, it follows that the (isomorphism class of the) representing object is an invariant of the category.

4.9.2 Definition

Let $\underline{N}$ be an additive category and $K \in$ Ob $\underline{N}$ such that $(\underline{N},K)$ is of type (a),(b) or (c). We define the trace group of $\underline{N}$, denoted by $T(\underline{N})$ to be $T(K)$.

From Theorem 1.5 we know that $Tr(\underline{N},T(\underline{N}) \cong Hom_{\underline{Z}}(T(\underline{N}),T(\underline{N}))$.

4.9.3 Definition

An element $t \in Tr(\underline{N},T(\underline{N}))$ corresponding under such an isomorphism to the identity map on $T(\underline{N})$ is called a universal trace.

As usual with universal objects, a universal trace is unique up to isomorphism, and be abuse of language we speak of the universal trace, and denote it by 'trace'.

4.9.4 Corollary

If $\underline{N}$ is an additive category of type (a),(b) or (c) (i.e. there exists a projective object K such that $(\underline{N},K)$ is of type (a),(b)or (c)) and $t = \{t_N : \text{End } N \longrightarrow G\}$ is a trace with values in a group G, then there is a unique (up to automorphism of $T(\underline{N})$) homomorphism $d_G : T(N) \longrightarrow G$ such that $t = d_G.\text{trace}$. ////.

4.9.5 The proof of the theorem in case (c) does not require the full strength of the conditions, but that the category $\underline{C}$ is a certain type of subcategory of an abelian category, and there is an important example of such a situation.

Let R be a commutative noetherian ring (with identity) and let R-mod be the category of R modules. Let C(R) be the full subcategory of those finitely generated R-modules which have finite homological dimension. Then the proof of the theorem in case (c) applies to C(R) although in general C(R) is not abelian (see appendix for an example) We just observe that the various constructions made in the proof all lie inside C(R). In particular, the kernels K_j in (4.8) belong to C(R) since there is a finite projective resolution by finitely generated objects:

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_{j+1} \longrightarrow K_j \longrightarrow 0$$

5 The theormm for S-linear categories.

5.1 Definition Let S be a commutative ring with identity. A category $\underline{N}$ is called an S-linear category if it is additive and if

(i) $\underline{N}(A,B)$ is an S-module for all $A,B \in \text{Ob } \underline{N}$

(ii) If $f:A \longrightarrow B$ is a map in $\underline{N}$ then the maps

$$f^* : \underline{N}(B,X) \longrightarrow N(\underline{A},X)$$
$$f_* : \underline{N}(X,A) \longrightarrow \underline{N}(B,X)$$

are S-module homomorphisms for all X.

Suppose G is an S module, then a trace $t = \{t_N : \text{End } N \longrightarrow G\}$ is called an S-trace if each t_N is an S-module homomorphism. The S-traces on $\underline{N}$ with values in G form a subgroup of $Tr(\underline{N},G)$ denoted by $Tr_S(\underline{N},G)$. It is an S-module in the obvious way. Theorem 1.5 can then be modified to give

5.2 Theorem

Let $\underline{N}$ be an S-linear category, and $K \in Ob\ N$ a projective object. If $(\underline{N},K)$ is of type (a), (b) or (c) then

$$\eta_K : Tr_S(\underline{N},G) \longrightarrow Hom_S(T(K),G)$$

is an S-module isomorphism. ////.

6 Examples

6.1 If F is a field, and $\underline{V(F)}$ is the category of finite dimensional vector spaces over F (type (a)) then $T(\underline{V(F)}) = F$, and the universal trace is the usual trace.

6.2 If $\underline{A}$ is the category if finitely generated abelian groups (type (c)) then $T(\underline{A}) = \mathbb{Z}$, and the universal trace is the usual trace.

6.3 If R is a commutative noetherian ring, and $\underline{C(R)}$ is the category introduced in(4.9.5), then $T(\underline{C(R)}) = R$.

6.4 If H is a finite group, let $\underline{H}$ be the category of finite dimensional complex representations of H. Let $Z_1, \ldots, Z_m$ be a complete set of distinct irreducible representations. From Schur's Lemma we know that

$$\begin{aligned} \underline{H}(Z_i, Z_j) &= 0 \text{ if } i \neq j \\ &= \mathbb{C} \text{ if } i = j \end{aligned}$$

Let $K = Z_1 \oplus Z_2 \oplus \cdots \oplus Z_m$. Then $\underline{H}$ is a $\mathbb{C}$-linear category and $(\underline{H},K)$ is of type (b). Then $T(\underline{H}) = \mathbb{C}$ is a $\mathbb{C}$-module, since

$$T(K) = T(Z_1 \oplus \cdots \oplus Z_m) = End_{\underline{H}}(Z_1) \oplus \cdots \oplus End_{\underline{H}}(Z_m)$$

If $p_i : T(\underline{H}) \longrightarrow \mathbb{C}$ is projection onto the i-th component, and $T^i = p_i . trace$ is the corresponding $\mathbb{C}$-trace, then T^i of a representation is roughly speaking the part of the character coming from Z_i. In particular, If $V \in Ob\ \underline{H}$, then $T^i(1_V)$ is an integer, and is the multiplicity of Z_i in V.

6.5 Let R be a commutative ring. Let $\underline{P(R)}$ be the category of finitely generated projective modules, and $\underline{H(R)}$ the category of those modules which have finite resolutions by finitely generated projective R modules. By similar remarks to (4.9.5) the theorem applies to $\underline{H(R)}$ although it may not be abelian. For these categories, $T(\underline{P(R)}) = R$ and $T(\underline{H(R)}) = R$.

If $P \in \text{Ob}\ \underline{P(R)}$ define $d(P) \in R$ to be trace (1_P). If $0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0$ is an exact sequence in $\underline{P(R)}$ then $d(P) = d(P') + d(P'')$, so that d defines a homomorphism

$$d:K(R) \longrightarrow R$$

where K(R) denotes the Grothendieck group of the category $\underline{P(R)}$. As an example of the theorem, we shall prove in the remainder of this section:

6.6 Theorem

The map $d:K(R) \longrightarrow R$ is a ring homomrphism whose image is the subring consisting of finite sums of idempotents.

6.7 Corollary

The trace of an idempotent matrix over a commutative ring is a sum of idempotents.

Proof(of 6.7). Let A be an n x n matrix over a commutative ring R such that $A^2 = A$. Then A corresponds to an R-module homomorphism

$$A:R^n \longrightarrow R^n \qquad \text{with } A^2 = A.$$

Since A is a projection operator, the image of A is a finitely generated projective R module, say P. But trace(A) = d(P) is a sum of idempotents by Theorem 6.6.

////.

We first show that d is a ring homomorphism.

6.8 Lemma

If $p:P \longrightarrow P$ and $q:Q \longrightarrow Q$ in $\underline{P(R)}$, then

$$\text{trace}(p \otimes q) = \text{trace}(p).\text{trace}(q)$$

Proof: Think of $q:Q \longrightarrow Q$ as being fixed for the time being. For $E \in \mathrm{Ob}\, P(R)$ and $e:E \longrightarrow E$ define

$$t_E(e) = \mathrm{trace}_{E \otimes Q}(e \otimes q) - \mathrm{trace}_E(e).\mathrm{trace}_Q(q)$$

The collection of t_E forms a trace on $P(R)$ with values in R, say t. But $\eta_R(t)(r) = t_R(r) = \mathrm{trace}_{R \otimes Q}(r \otimes q) - \mathrm{trace}_R(r).\mathrm{trace}_Q(q)$

$= 0$ since the universal trace is an R-trace.

Hence $t = 0$, since η_R is injective, so that

$$t_P(p) = 0 = \mathrm{trace}(p \otimes q) - \mathrm{trace}(p).\mathrm{trace}(q). \qquad ////.$$

Since $1_{P \otimes Q} = 1_P \otimes 1_Q$, it follows that $d:K(R) \longrightarrow R$ is a ring homomrphism.

Next we consider the effect on the trace of localisatiom. Let S be a multiplicatively closed subset of R, and let $S^{-1}R$ be the associated ring of fractions. Let $s:R \longrightarrow S^{-1}R$ be the natural map.

6.9 Lemma

If $P \in \mathrm{Ob}\, \underline{P(R)}$ then the following diagram commutes:

$$\begin{tikzcd}
\mathrm{End}_R(P) \arrow[r] \arrow[d, dashed, "S^{-1}"'] & R \arrow[d, "s"] \\
\mathrm{End}_{S^{-1}R}(S^{-1}P) \arrow[r] & S^{-1}R
\end{tikzcd}$$

where as usual 'trace' denotes the universal trace in the corresponding category.

Proof: Since $S^{-1}:\underline{P(R)} \longrightarrow \underline{P(S^{-1}R)}$ is exact, we have

$$\mathrm{trace}.S^{-1} \in \mathrm{Tr}(\underline{P(R)}, S^{-1}R)$$

But we also have

$$a.\mathrm{trace} \in \mathrm{Tr}(P(K), S^{-1}R)$$

In order to prove these two traces are equal, we only have to look at them on R, which means checking the commutativity of the diagram

6.10 Remark The analogous result holds for the category $\underline{H}(R)$ (see (6.5)).

6.11 Corollary

If R *is an integral domain and* K *its field of fractions, and if* $f:M \longrightarrow M$ *in* $\underline{H}(R)$ (or $\underline{P}(R)$) *then*

$$\text{trace } f = \text{trace } (1 \otimes_R f) \qquad (1 \otimes_R f : K \otimes_R M \to K \otimes_R M)$$

(where we think of $R \subseteq K$)   ////.

6.12 Remark This is the fact one uses to calculate the trace of an endomorphism of a finitely generated abelian group.

6.13 Corollary

If $\underline{p} \subset R$ *is a prime ideal, and* $f:P \longrightarrow P$ *in* $\underline{P}(R)$, *then*

$$(\text{trace } f)_{\underline{p}} = \text{trace } (f_{\underline{p}}) \in R_{\underline{p}}$$

where the subscript $\underline{p}$ *denotes localisation at the prime ideal.*   ////.

6.14 Let $r:K(R) \longrightarrow H_0(R)$ be the augmentation map of algebraic K-theory ((4),p 458). We define a map $u:H_0(R) \longrightarrow R$ as follows: Suppose $f:\text{spec } R \longrightarrow \mathbb{Z}$ is continuous, and let $U_n = f^{-1}\{n\}$. Then U_n is an open and closed subspace of spec R, so there is a unique idempotent $e_n \in R$ such that

$$U_n = \{ \underline{p} \in \text{spec } R : \underline{p} \not\supseteq Re_n \}$$

(see (5),Theorem 7.12). Define $u(f) = \sum_n ne_n \in R$. This is a finite sum since only a finite number of U_n are non-empty. Clearly $u:H_0(R) \longrightarrow R$ is a homomorphism, is natural and its image is the subring of R consisting of finite sums of idempotents. The proof of Theorem 6.6 follows immediately from the following proposition:

6.15 Proposition

The diagram

$$\begin{tikzcd} K(R) \arrow[rr, "d"] \arrow[dr, "r"'] & & R \\ & H_0(R) \arrow[ur, "u"'] & \end{tikzcd}$$

commutes.

Proof: We first observe that the diagram commutes when R is a local ring, for then if $P \in \underline{P}(R)$, then P is free of rank $r(P)$, $(H_0(R) = \mathbb{Z})$ and $ur(P) = r(P).1 = d(P).1$.

In the general case, let $P \in \mathrm{Ob}\ \underline{P}(R)$, and let $a(P) = d(P) - ur(P)$ Multiplication by $a(P)$ defines an R module homomorphism

$$a(P) : R \longrightarrow R.$$

Suppose $\underline{p} \in \mathrm{spec}\ R$. Then localisation of the above map defines an $R_{\underline{p}}$ module homomorphism

$$a(P)_{\underline{p}} : R_{\underline{p}} \longrightarrow R_{\underline{p}}$$

But by naturality of d,u,r, it follows that $a(P)_{\underline{p}} = a(P_{\underline{p}}) = 0$ by the first remark. The map $a(P)$ is thus locally zero at each point of spec R, so is zero. Hence $d(P) = ur(P)$, and $d = ur$. /////.

7. The topological version

7.1 Definition Let S be a topological ring (commutative with identity). A category $\underline{N}$ is a topological S-linear category if it is additive and if

(i) $\underline{N}(A,B)$ is a topological S-module for all $A,B \in \mathrm{Ob}\ \underline{N}$

(ii) If $f : A \longrightarrow B$ in $\underline{N}$ then the induced maps

$$f^* : \underline{N}(B,X) \longrightarrow \underline{N}(A,X)$$

$$f_* : \underline{N}(X,A) \longrightarrow \underline{N}(X,B)$$

are continuous S-module homomorphisms.

If $\underline{N}$ is a topological S-linear category, and S-trace $t = \{t_N : \mathrm{End}\ N \longrightarrow G\}$, where G is a topological S-module is called a continuous trace if each t_N is continuous. The continuous S-traces form an S-submodule $\mathrm{CtsTr}_S(\underline{N},G) \subseteq \mathrm{Tr}_S(\underline{N},G)$. By definition, if $K \in \mathrm{Ob}\ \underline{N}$ then $\eta_K \mathrm{CtsTr}_S(\underline{N},G) \subseteq \mathrm{CtsHom}_S(T(K),G)$, where $T(K)$ has the identification topology. When $(\underline{N},K)$ is of type (a) or (b) the reverse inclusion can be shown, by following the proof of Theorem 1.5 and checking continuity at each stage. Hence we have:

7.2 Theorem

If $\underline{N}$ is a topological S-linear category, and K Ob $\underline{N}$ is a projective object such that $(\underline{N},K)$ is of type (a) or (b) then

$$\eta_K : \mathrm{CtsTr}_S(\underline{N},G) \longrightarrow \mathrm{CtsHom}_S(T(K),G)$$

is an S-module isomorphism. ////.

8. Example

8.1 Let X be a compact hausdorff space, and let $\underline{\mathrm{Vect}(X)}$ be the category of finite dimensional complex vector bundles over X. Then $\underline{\mathrm{Vect}\ (X)}$ is a topological $\mathbb{C}$-linear category. Let $\mathbb{C}(X)$ be the ring of complex valued functions on X with the sup.norm topology. For if $E \in \mathrm{Ob}\ \underline{\mathrm{Vect}\ (X)}$ and we choose a metric on E, we can give $\Gamma(E)$, the space of sections of E, a topology making it a normed vector space, and this topology is independent of the choice of metric. Now if $E,F \in \mathrm{Ob}\ \underline{\mathrm{Vect}\ X}$, we know that

$$\underline{\mathrm{Vect}(X)}\ (E,F) \cong \Gamma(\mathrm{Hom}(E,F))$$

as complex vector spaces. We give $\underline{\mathrm{Vect}(X)}(E,F)$ the topology to make this a homeomorphism .

8.2 Proposition

There is a natural bijection between the set of complex regular Borel measures on X and the set $\mathrm{CtsTr}_{\mathbb{C}}\ (\underline{\mathrm{Vect}(X)}, \mathbb{C})$ where the trace t corresponds to the measure μ by means of the equation

$$t_E(f) = \int_X \mathrm{trace}(f_x)\ d\mu$$

where $f_x : E_x \longrightarrow E_x$ is the restriction of f to the fibre over x, and trace denotes the universal trace for complex vector spaces.

Proof: Let I denote the trivial line bundle over X, which is projective in $\underline{\mathrm{Vect}(X)}$. Identifying $\mathrm{End}_{\underline{\mathrm{Vect}(X)}}(I)$ with $\mathbb{C}(X)$, we know from Theorem 7.2 that

$$\eta_I : \mathrm{CtsTr}_{\mathbb{C}}\ (\mathrm{Vect}(X), \mathbb{C}) \longrightarrow \mathrm{CtsHom}_{\mathbb{C}}\ (\mathbb{C}(X), \mathbb{C})$$

By the Riesz Representation Theorem, the elements of $\mathrm{CtsHom}_{\mathbb{C}}(\mathbb{C}(X),\mathbb{C})$ correspond bijectively with complex regular Borel measures on X, and combining this correspondence with η_I yields the required bijection where $t = \{\mathrm{End}\ E \longrightarrow \mathbb{C}\}$ corresponds to μ by means of the equation

$$t_I(g) = \int_X g(x)d\mu \quad \text{for } g \in \mathbb{C}(X).$$

Let $T = \{\mathrm{End}\ E \longrightarrow \mathbb{C}(X)\}$ be the universal trace. Then $t = t_I.T$ (i.e. $t_E = t_I.T_E:\mathrm{End}\ E \longrightarrow \mathbb{C}(X) \longrightarrow \mathbb{C}$) since these two traces agree on the object I and so by Theorem 1.5 are equal.

Suppose $x \in X$, and $E \in \mathrm{Ob}\ \underline{\mathrm{Vect}\ X}$. If $f:E \longrightarrow E$ in $\underline{\mathrm{Vect}\ (X)}$ define $s_E(f) = T_E(f)(x) - \mathrm{trace}(f_x)$.
Then the collection of s_E defines a trace on $\underline{\mathrm{Vect}\ (X)}$ with values in $\mathbb{C}$. But this trace is zero on I, so s is the zero trace, i.e.

$$T_E(f)(x) = \mathrm{trace}(f_x).$$

Hence t corresponds to μ by the equation

$$t_E(f) = t_I.T_E(f) = \int_X T_E(f)(x)d\mu = \int_X \mathrm{trace}(f_x)d\mu \qquad ////.$$

9. The trace group of an additive category.

In this section we associate to an additive category an abelian group, together with a trace taking values in this group which is universal for traces and extends the definition given in 4.9.2.

9.1 Definition

Let $\underline{N}$ be an additive category. The trace group of $\underline{N}$, denoted by $T(\underline{N})$is an abelian group together with a trace $t \in \mathrm{Tr}(\underline{N},T(\underline{N}))$ which is universal with respect to traces. This means that for any trace say t' on $\underline{N}$ with values in an abelian group say G there is a unique homomorphism $f:T(N) \longrightarrow G$ such that $t' = f.t$.

To show that such a group exists, let $F(\underline{N})$ be the free abelian group on the pairs (M,f) where $M \in \mathrm{Ob}\ \underline{N}$ and $f \in \mathrm{End}\ M$. Let $S(\underline{N})$be the subgroup generated by elements of the form

(i) $(M,f) + (M,g) - (M,f+g)$ for all $M \in \operatorname{Ob} \underline{N}$, $f,g \in \operatorname{End} M$

(ii) $(E,e) + (G,g) - (H,h)$ for all commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r] \arrow[d, "e"'] & H \arrow[r] \arrow[d, "h"'] & G \arrow[r] \arrow[d, "g"'] & 0 \\ 0 \arrow[r] & E \arrow[r] & H \arrow[r] & G \arrow[r] & 0 \end{tikzcd}$$

with exact rows

(iii) $(M,fg) - (M,gf)$ for all $M \in \operatorname{Ob} \underline{N}$, $f,g \in \operatorname{End} M$.

Then let $T(\underline{N}) = F(\underline{N})/S(\underline{N})$ and let $t \in \operatorname{Tr}(\underline{N}; T(\underline{N}))$ be the trace defined on the object M by

$$t_M(f) = [M,f] \quad \text{for } f \in \operatorname{End} M$$

where $[M,f]$ denotes the element of $T(\underline{N})$ which is the image of (M,f) in $F(\underline{N})$. Clearly t is the universal trace.

We can think of Theorem 1.5 as calculating the trace group of those categories to which the theorem applies.

Appendix

If R is a commutative ring with identity, let $\underline{P(R)}$ be the category of finitely generated projective R-modules, and $\underline{H(R)}$ the category of those finitely generated R-modules which have finite resolutions by objects in $\underline{P(R)}$.

Clearly if R is a noetherian ring such that each finitely generated module has finite homological dimension (although this may not be bounded) then $\underline{H(R)}$ is the category of all finitely generated R-modules, and so is abelian.

We now give an example of a noetherian local ring R for which a certain map in $\underline{H(R)}$ does not have a kernel. (We must be careful not to assume that the categorical kernel is the usual kernel, or even a submodule of the usual kernel)

Consider the ring Z_4 of integers modulo 4, whose elements we write as $\{\hat{0},\hat{1},\hat{2},\hat{3}\}$. Then Z_4 is noetherian and has a unique maximal ideal, namely $A = \{\hat{0},\hat{2}\}$. Let $m: Z_4 \longrightarrow Z_4$ be multiplication by $\hat{2}$. Then A is the image of m, and since there does not exist a Z_4 module map $j: A \longrightarrow Z_4$ such that $mj = 1$, we know that A is not projective. If A had finite homological dimension, say n, then the projective resolution

$$\underbrace{Z_4 \xrightarrow{m} Z_4 \xrightarrow{m} Z_4 \xrightarrow{m} \quad\cdots\quad \xrightarrow{m} Z_4}_{n \text{ times}} \xrightarrow{m} A \longrightarrow 0$$

would imply that the (usual) kernel of the first map is projective. But this is A, so A must have infinite homological dimension, so $A \notin \mathrm{Ob}\ \underline{H(Z_4)}$.

Now $m: Z_4 \longrightarrow Z_4$ is a map in $\underline{H(Z_4)}$, and we show that this map has no kernel. Assume for a contradiction that $i: K \longrightarrow Z_4$ is the kernel. Since $mi = 0$, i factors through A by a unique map $j: K \longrightarrow A$. But A is an irreducible Z_4 module, so either $j = 0$, or j is surjective.

Since $mm = 0$, and m is a map in $\underline{H(Z_4)}$, there is a unique map $h: Z_4 \longrightarrow K$ such that $ih = m$.

$$\begin{tikzcd}
K \arrow[rr, "i"] \arrow[dr, dashed, "j"] & & Z_4 \arrow[r, "m"] & Z_4 \\
& A \arrow[ur, dashed] & & \\
& & Z_4 \arrow[ull, dashed, "h"] \arrow[ul] \arrow[uu, "m"'] &
\end{tikzcd}$$

But $m \neq 0$, so $i \neq 0$, so $j \neq 0$. Hence j is surjective.

On the other hand, j is injective. It suffices to prove that i is injective. Suppose that $i(k) = 0$ for some $k \in K$. Let $f: Z_4 \longrightarrow K$ be the map $f(n) = nk$. Then f is a map in $\underline{H(Z_4)}$ and $if = 0$. But this implies that $i(h + f) = ih + if = ih = m$, and so by uniqueness of h, $h + f = h$, so $f = 0$. Hence $k = 0$, so i is injective.

Thus $j: K \cong A$, But A has infinite homological dimension, while K has finite homological dimension, which is a contradiction, so $m: Z_4 \longrightarrow Z_4$ has no kernel in $\underline{H(Z_4)}$.

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CHAPTER 5 : INTRODUCTION TO THE THEORY OF CHARACTERISTIC CLASSES IN ALGEBRAIC K-THEORY

INTRODUCTION TO THE THEORY OF CHARACTERISTIC CLASSES IN ALGEBRAIC K-THEORY

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If E is a complex vector bundle over a compact hausdorff space X, there are associated characteristic classes $c_i(E) \in H^{2i}(X)$ which can be thought of as obstructions to E either containing or being contained in a trivial bundle of some dimension. Motivated by the important theorem of Swan (6), the problem arises to define and investigate characteristic classes for finitely generated modules over a commutative ring. These can then be interpreted as obstructions to the module having a certain number of linearly independent or generating elements. A very special case of this was considered by Ozeki (9) in terms of de Rham cohomology.

The approach of this chapter is motivated by Atiyah's work on operations in (topological) K-theory (7), which makes use of the notion of a λ-ring, intriduced by Grothendieck (10), and the γ-operations arising from an augmented λ-ring. It is well known (see for example Tall & Atiyah (8)) that if A is a commutative ring with identity, then K(A) is a λ-ring, and that if one chooses a prime ideal $\underline{p} \subset A$, then $K(A_{\underline{p}}) \cong \mathbb{Z}$ and the natural map $K(A) \longrightarrow K(A_{\underline{p}})$ provides an augmentation, enabling one to introduce the γ-operations.

The two disadvantages of studying operations from this approach are are first that this depends on a choice of prime ideal , and second that this choice neglects some of the structure of the ring arisng from the other prime ideals. We overcome these difficulties by considering all prime ideals at once via the ring $H^0(A)$ (see 0.10). This makes the constructions natural, and allows us to consider the topological theory as a subset of the algebraic theory.

The chapter contains two main theorems as follows.

4.14 Theorem *There is a split monomorphism*

$$w:\mathbb{Z}[[s_1,s_2,\ldots]] \longrightarrow \mathrm{Op}(Rk_0,K).$$

Again motivated by topological K-theory and results of Adams & Atiyah(12) we introduce chern classes taking values in a graded ring $H^*(A)$, with the property that:

5.6 Theorem

$$c_1:\mathrm{Pic}\ (A) \longrightarrow H^1(A) \text{ is an isomorphism.}$$

There is also an important decomposition principle (2.12) for algebraic K-theory derived from a particular interpretation of $H^0(A)$ in terms of 'blocks' introduced in § 2.

The sections are as follows.

0. Preliminaries.
1. The ring of continuous functions.
2. Partitions.
3. Operations in Algebraic K-theory.
4. Dimension properties.
5. The graded ring and relations with the Picard group.
6. Appendix.

Section 0 is expository, and section 1 is a categorical presentation of well known ideas. Apart from the above references, the work is original although it has since been discovered that similar work has been done by Berthelot (12) (unpublished).

0. Preliminaries.

0.1 Unless otherwise stated, all rings in this chapter w ll be commutative and have an identity, and all ring homomorphisms will preserve the identity. All modules will be unital. The category of rings will be denoted by $\underline{R}$. If A is a ring, let

$\underline{M}(A)$ be the category of finitely generated A-modules

$\underline{P}(A) \subseteq \underline{M}(A)$ be the category of finitely generated projective A-modules.

0.2 Given a ring homomorphism $f: A_1 \longrightarrow A_2$, we may consider A_2 as an A_1-module and this induces functors

$$f_*: \underline{M}(A_1) \longrightarrow \underline{M}(A_2)$$
$$f_*: \underline{P}(A_1) \longrightarrow \underline{P}(A_2)$$

defined by $M \longmapsto A_2 \otimes_{A_1} M$. If $g: A_2 \longrightarrow A_3$ is a second ring homomorphism then $(gf)_* = g_* f_*$ and $1_* = 1$.

0.3 If $A \in \mathrm{Ob}\, \underline{R}$, let $K(A)$ denote the Grothendieck group of the category $\underline{P}(A)$. Tensor product of modules induces a multiplication in $K(A)$ which makes $K(A)$ into a commutative ring with identity. By 0.2, a map $f: A_1 \longrightarrow A_2$ in $\underline{R}$ induces a ring homomorphism

$$f_*: K(A_1) \longrightarrow K(A_2)$$

in a natural way, i.e. K is a functor from R to R.

0.4 Definition A ring A is a local ring if it has a unique maximal ideal.

0.5 Notation We shall usually denote ideals by small letters underlined, for example $\underline{a}, \underline{p}$.

0.6 If A is a ring and $\underline{p} \subset A$ is a prime ideal, we write $A_{\underline{p}}$ for the localisation of A at $\underline{p}$. Then $A_{\underline{p}}$ is a local ring whose maximal ideal is the image of $\underline{p}$ under the natural map $A \longrightarrow A_{\underline{p}}$. If M is an A-module, we write $M_{\underline{p}}$ for the local sation of M at $\underline{p}$, where $M_{\underline{p}}$ is the $A_{\underline{p}}$-module $A_{\underline{p}} \otimes_A M$. This construction defines a functor

$$\underline{M}(A) \longrightarrow \underline{M}(A_{\underline{p}})$$

which is exact (1).

0.7 Proposition

If M,N are A-modules, and $f:M \longrightarrow N$ is an A map then

(i) $M = 0$ *iff* $M_{\underline{p}} = 0$ *for all prime ideals* $\underline{p} \subset A$.

(ii) *f is injective (resp. surjective) iff* $f_{\underline{p}}:M_{\underline{p}} \longrightarrow N_{\underline{p}}$ *is injective (resp. surjective) for all prime ideals* $\underline{p} \subset A$.

Proof: See Atiyah & Macdonald (1) Prop.3.5 and Prop.3.6. ////.

0.8 If $A \in \mathrm{Ob}\ \underline{R}$, let spec A be the set of prime ideals with the Zariski topology. More precisely, if $\underline{a}$ is an ideal of A, let

$$W(\underline{a}) = \{\underline{p} \in \text{spec } A : \underline{p} \not\supset \underline{a}\} .$$

Then these subsets of spec A satisfy

(i) $W(0) = \emptyset$ and $W(A) = \text{spec } A$

(ii) $W(\sum_i \underline{a}_i) = \bigcup_i W(\underline{a}_i)$

(iii) $W(\underline{a}) \cap W(\underline{b}) = W(\underline{a} \cap \underline{b}) = W(\underline{ab})$

(iv) $\underline{a} \subseteq \underline{b} \Rightarrow W(\underline{a}) \subseteq W(\underline{b})$

We give spec A the topology which has the $W(\underline{a})$ as open sets. A map $f:A_1 \longrightarrow A_2$ in $\underline{R}$ induces in a natural way a function

$$f^*:\text{spec } A_2 \longrightarrow \text{spec } A_1$$

by the rule $f^*(\underline{p}) = f^{-1}(p)$. In fact f^* is continuous with respect to the Zariski topologies, and makes spec into a functor from $\underline{R}$ to the topological category. In general spec A is not Hausdorff, but it is quasi-compact ((2) Cor.7.3).

0.9 Proposition

There is a Bijective correspondence between:

(i) open and closed subsets of spec A

(ii) idempotents of A.

The idempotent e corresponds to the open and closed set $W(\underline{Ae})$.

Proof: See Swan ((2) Theorem 7.12). ////.

<u>0.10</u> Define $H^0(A) = \{\text{continuous maps spec } A \longrightarrow \mathbb{Z}\}$ where $\mathbb{Z}$ has its usual (discrete) topology. Clearly H^0 is a functor from $\underline{R}$ to $\underline{R}$.

<u>0.10.1</u> If $P \in \text{Ob } \underline{P}(A)$ and $\underline{p} \in \text{spec } A$, then $P_{\underline{p}} \in \text{Ob } P(A_{\underline{p}})$ is a finitely generated projective module over a local ring, so by a well known theorem (see for example(5), is free, and so

$$P_{\underline{p}} \cong (A_{\underline{p}})^{r_P(\underline{p})} \qquad \text{where } r_P(p) \in \mathbb{Z}$$

<u>0.10.2 Proposition</u>

(i) <u>The function</u> $r_P : \text{spec } A \longrightarrow \mathbb{Z}$ <u>is continuous</u>.

(ii) $r_{P \oplus Q} = r_P + r_Q$

(iii) $r_{P \otimes Q} = r_P . r_Q$

(iv) <u>If</u> $\lambda^i P$ <u>denotes the i-th exterior power of</u> P, <u>then</u>

$$r_{\lambda^i P} = \binom{r_P}{i} \quad \text{(\underline{binomial coefficient})}.$$

<u>Proof</u> (i) see ((2),p136).

(ii),(iii),(iv) see ((4), page 142). ////.

<u>0.10.3</u> If $P \in \text{ob } \underline{P}(A)$, then by Prop.0.10.2, the function $r_P : \text{spec } A \longrightarrow \mathbb{Z}$ is continuous, Then $U_n = r_P^{-1}\{n\}$ is an open and cloeed subset of spec A, so there is a unique idempotent e_n such that $U_n = W(Ae_n)$ by

Let r_P denote the A-module

$$r_P = \bigoplus_n (Ae_n)^n$$

(Since spec A is quasi-compact, only a finite number of U_n are non-empty, and so this is a finite sum.)

<u>0.11 Deiinition</u> A λ-ring is a commutative ring A with identity together with a family $\{\lambda^n : n = 0,1,\ldots\}$ of functions

$$\lambda^n : A \longrightarrow A$$

satisfying the following three conditions:

(i) $\lambda^0(x) = 1$ for all $x \in A$

(ii) $\lambda^1(x) = x$ forlall $x \in A$

(iii) $\lambda^n(x + y) = \sum_{i+j=n} \lambda^i(x). \lambda^j(y)$ for all $x,y \in A$, all $n \geqslant 0$.

0.12 A λ-map between two λ-rings is a ring homomorphism which commutes with the λ-operations. If A,B are λ-rings the direct product A x B inherits a λ-ring structure by the rule

$$\lambda^n(a,b) = (\lambda^n a, \lambda^n b)$$

Moreover the projections $A \times B \longrightarrow A$ and $A \times B \longrightarrow B$ are λ-maps. [In fact this makes A x B into the product of A and B in the category of λ-rings.]

0.13 Definition A λ-ring is normal if $\lambda^n(1) = 0$ for $n \geq 2$.

0.14 Let $A \in \mathrm{Ob}\ \underline{R}$. Let $A[[t]]$ denote the ring of power series in t, and let $1 + A[[t]]^+$ denote the subset of elements with constant term 1 Then $1 + A[[t]]^+$ is an abelian group if we define composition to be multiplication of power series. We write this operation ' $\oplus$ ' , and s'

$$(1 + \sum a_i t^i) \oplus (1 + \sum' b_i t^i) = (1 + \sum c_i t^i)$$

where

$$c_n = \sum_{i+j=n} a_i b_j \ .$$

0.14 If A is a λ-ring, defint $\lambda_t : A \longrightarrow 1 + A[[t]]^+$ by

$$\lambda_t(a) = \sum_{n \geq 0}' \lambda^n(a) t^n = 1 + at + \sum_{n \geq 2} \lambda^n(a) t^n \ .$$

Then λ_t is a homomorphism of the underlying abelain groups, i.e.

$$\lambda_t(a + b) = \lambda_t(a) \oplus \lambda_t(b)$$

If A is normal, then $\lambda_t(1) = 1 + t$, so $\lambda_t(n) = (1 + t)^n$. Hence there is a unique normal λ-ring structure on $\mathbb{Z}$ given by

$$\lambda^n(m) = m(m-1)\dots(m-n+1) \quad = \binom{m}{n}$$

0.15 If A is a λ-ring, define

0.15.1 $$\gamma^n(a) = \lambda^n(a + n - 1)$$

0.15.2 $$\gamma_t(a) = \sum_{n \geq 0} \gamma^n(a) t^n \in A[[t]]$$

0.16 Proposition

Let A be a normal λ-ring. Then

(i) $\gamma^0(a) = 1$, $\gamma^1(a) = a$, $\gamma^n(x+y) = \sum_{i+j=n} \gamma^i(x)\, \gamma^j(y)$

(ii) $\gamma_t(x) = \lambda_{t/1-t}(x)$ and $\lambda_t(x) = \gamma_{t/1+t}(x)$.

Proof. It suffices to prove the first part of (ii).

$$\lambda_{t/1-t}(x) = \sum_n \lambda^n(x)t^n(1-t)^{-n}$$
$$= \sum_n \lambda^n(x) \sum_{m \geq n} \binom{m-1}{m-n} t^m$$

The coefficient of t^m is

$$= \sum_{n=0}^{m} \lambda^n(x) \binom{m-1}{m-n} = \sum_{n=0}^{m} \lambda^n(x) \quad \lambda^{m-n}(m-1)$$

$$= \lambda^m(x+m-1) = \gamma^m(x).$$

////.

0.17 Theorem .

If $A \in$ Ob $\underline{R}$ then

(i) the exterior power functors λ^n:P(A) $\longrightarrow$ P(A) induce a λ-ring structure on K(A), which is normal.

(ii) the functions $\lambda^n:\mathbb{Z} \longrightarrow \mathbb{Z}$ which define on $\mathbb{Z}$ the normal λ-ring structure (see 0.14) induce by composition fünctions $\lambda^n:H^0(A) \longrightarrow H^0(A)$ whish define a λ-ring structure on $H^0(A)$ which is normal.

(iii) the function P(A) $\longrightarrow H^0(A)$ which sends P to r_P (see 0.10) induces a λ-map $r:K(A) \longrightarrow H^0(A)$.

(iv) there is a natural map $i:H^0(A) \longrightarrow K(A)$ such that $ri = 1$, and i is a λ-map.

Proof:

(i) For $P \subset$ Ob P(A), define $\lambda_t(P) = \sum_{n \geq 0} \lambda^n(P)t^n \quad 1 + K(A)[[t]]^+$.

hen λ_t is additive, so factors through a map $\lambda_t:K(A) \longrightarrow 1 + K(A)[[t]]$
Let $\lambda^n:K(A) \longrightarrow K(A)$ be the coefficient of t^n. These operations defiee the λ-ring structure, which is normal since $\lambda^n(A) = 0$ for $n \geq 2$, and A is the identity in K(A).

(ii) Since the λ-operations are defined by composition, it is clear that $H^0(A)$ is a normal λ-ring since the integers is (see o.14)

(iii) That $r:K(A) \longrightarrow H^0(A)$ is a ring homomorphism follows trivially from Prop.0.10.2. We prove that it commutes with the λ-operations.

An element of $K(A)$ can be written $P - n$ where n denotes A^n. Since $K(A)$ is normal, $\lambda_t(n) = (1+t)^n$, so

$$\lambda_t(-n) = (1 + t)^{-n} = \sum_{m \geqslant 0} a_m t^m \quad \text{say.}$$

Then $\lambda^k(P - n) = \sum_{i+j=k} \lambda^i(P)\,\lambda^j(-n) = \sum_{i+j=k} \lambda^i(P) a_j$

so $r\,\lambda^k(P - n) = \sum_{i+j=k} r\,\lambda^i(P) a_j$

since r is a ring homomorphism and $r(a_j) = a_j$.

Hence

$$r\,\lambda^k(P - n) = \sum_{i+j=k} \lambda^i(r_P) a_j \qquad \text{by } 0.10.2$$

$$= \lambda^k(r_P - n) \quad \text{since } H^0(A) \text{ is normal.}$$

i.e. $r\,\lambda^k = \lambda^k r$

(iv) If $f:\text{spec } A \longrightarrow \mathbb{Z}$ is continuous and takes non-negative values, then for each $n \geqslant 0$, let $U_n = f^{-1}\{n\}$ which is an open and closed subspace of spec A. Thus there exists a unique idempotent e_n such that $U_n = W(Ae_n)$. Let $i(f) = \bigoplus_n (Ae_n)^n$. If $h:\text{spec } A \longrightarrow \mathbb{Z}$ is any continuous function we may write it as $f - g$ where f,g take non-negative values, and define $i(h) = i(f) - i(g)$. This is well defined (see (5)pp459-460). The proof of the proposition is then similar to (iii) above.

////.

<u>0.18</u> Since $i:H^0(A) \longrightarrow K(A)$ is injective, we shall identify $H^0(A)$ with its image. For $a \in K(A)$, we write $[a]$ for $a - ir(a)$.

If $A,B \in \text{Ob } \underline{R}$, let $p:A \times B \longrightarrow A$ and $q:A \times B \longrightarrow B$ be the projections. Then p induces $p^*:\text{spec } A \longrightarrow \text{spec } (A \times B)$ and $p_*:K(A \times B) \longrightarrow K(A)$.

<u>0.19 Theorem</u>

(i) (p^*,q^*): spec A $\sqcup$ spec B $\longrightarrow$ spec $(A \times B)$ <u>is a homeomorphism</u>.

(ii) (p_*,q_*): $H^0(A \times B) \longrightarrow H^0(A) \times H^0(B)$ <u>is an isomorphism of λ-rings</u>.

<u>(iii)</u> (p_*,q_*): $K(A \times B) \longrightarrow K(A) \times K(B)$ <u>is an isomorphism of λ-rings</u>.

<u>(iv) The isomorphisms of (ii) and (iii) commute with the maps</u> i,r <u>of Theorem 0.17</u>.

<u>Proof</u>:

(i) See (2)pp . (ii) follows from (i).

(iii) If $P \in$ Ob $\underline{P(A)}$ and $Q \in$ Ob $\underline{P(B)}$ then $P \times Q$ is an $A \times B$ module and moreover is projective. This defines a functor

$$\underline{P(A)} \times \underline{P(B)} \longrightarrow \underline{P(A \times B)}$$

which induces a map $K(A) \times K(B) \longrightarrow K(A \times B)$. We show that this is the inverse to (p_*,q_*).

Suppose $P \in \underline{P(A)}$ and $Q \in \underline{P(B)}$. Then $p_*(P \times Q) \cong P$. For, the map $p_*(P \times Q) = A \otimes_{A \times B}(P \times Q) \longrightarrow P$ given by

$$a \otimes (p,q) \longmapsto ap$$

is an A-module isomorphism with inverse given by

$$p \longmapsto 1 \otimes (p,0).$$

This shows that the map is a right inverse for (p_*,q_*)

Conversely, suppose $P \in \underline{P(A \times B)}$. Then $p_*P \times q_*P \cong P$, since the map $p_*P \times q_*P = (A \otimes_{A \times B} P) \times (B \otimes_{A \times B} P) \longrightarrow P$ given by

$$(a \otimes p_1 , b \otimes p_2) \longmapsto (a,0)p_1 + (0,b)p_2$$

is an $A \times B$ module isomorphism with inverse

$$p \longmapsto ((1,0) \otimes p , (0,1) \otimes p)$$

This shows the map is a left inverse, which completes the proof.

(iv) follows by naturality of the isomorphisms.

////.

In section 4, we shall need the concept of 'formal power series' in an infinite set of variables, and in order to be precise, we describe this ring in some detail.

<u>0.20</u> Let $R \in \mathrm{Ob}\ \underline{R}$, and let A be a set. Let N denote the non-negative integers, and for each $a \in A$, let N_a be a copy of N. Let

<u>0.20.1</u>
$$N_A = \bigoplus_{a \in A} N_a$$

i.e. the set of sequences $\{n_a : n_a \in N_a \ \&\ n_a = 0$ for all but a finite number of $a\}$. Then N_A is a commutative semigroup with the property that given $s \in N_A$, there is only a finite set of solutions of the equation

$$x + y = s.$$

Consider the set of functions $f : N_A \longrightarrow R$, which we denote by $R(A)$. If $f, g \in R(A)$, define

<u>0.20.2</u>
$$(f + g)(s) = f(s) + g(s)$$
$$(fg)(s) = \sum_{x+y=s} f(x)g(y) .$$

Then $R(A)$ is a commutative ring, and we may identify $r \in R$ with the element $r : N_A \longrightarrow R$ given by

<u>0.20.3</u>
$$\begin{aligned} r(s) &= r \quad \text{if } s = 0 \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

An element $f : N_A \longrightarrow R$ is a polynomial if the set

$$\{s \in N_A : f(s) \neq 0\} \quad \text{is finite.}$$

Let $e : N_A \longrightarrow N$ be the augmentation map $\{n_a\} \longmapsto \sum n_a$. If f is a polynomial, we define

<u>0.20.4</u>
$$\text{degree of } f = \deg (f) = \max \{e(s) \text{ such that } f(s) \neq 0\}$$

A polynomial h is homogeneous of degree n if $h(s) = 0$ whenever $e(s) \neq \deg(h) = n$. If f is a polynomial, define

<u>0.20.5</u>
$$\begin{aligned} f_r(s) &= f(s) \text{ if } e(s) = r \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then f_r is homogeneous of degree r, and $f = \sum_{r=0}^{\deg f} f_r$.

Let $R[A] \subseteq R(A)$ be the set of polynomials:it is clearly a subring. Guided by the idea that a power series is roughly speaking a 'polynomial of infinite degree' we want a power series to be a formal sum

$$\sum_{r=0}^{\infty} f_r$$

where f_r is a homogeneous polynomial of degree r. Precisely, a function $f \in R(A)$ is a formal power series if for each $r \in N$, the function $f_r \in R(A)$ defined analogously to 0.20.5 is a polynomial.

The set of formal power series forms a subring denoted by $R[[A]]$ which contains $R[A]$.

<u>0.20.6 Remark</u> If A is a finite set, then $R(A) = R[[A]]$, and so one might take $R(A)$ as the appropriate generalisation of 'formal power series' in a finite number of variables. However, for the application in this chapter we require the extra finiteness condition described above.

1. The ring of continuous functions.

1.1 Let $\underline{C}$ be the category of compact hausdorff spaces, and let $\underline{C}_{con} \subset \underline{C}$ be the full subcategory of connected spaces. For $X \in \mathrm{Ob}\,\underline{C}$ let $\mathbb{C}(X)$ be the ring of continuous complex valued functions on X. If $f: X \longrightarrow Y$ in $\underline{C}$, then f induces by composition a ring homomorphism

$$\mathbb{C}(f): \mathbb{C}(Y) \longrightarrow \mathbb{C}(X) \qquad g \longmapsto gf$$

in a natural way, so that $\mathbb{C}$ is a functor from $\underline{C}$ to $\underline{R}$.

1.2 Proposition

$\mathbb{C}: \underline{C} \longrightarrow \underline{R}$ is a (contravariant) embedding.

Proof: We have to show that if $f,g: X \longrightarrow Y$ and $f \neq g$, then $\mathbb{C}(f) \neq \mathbb{C}(g)$. Let $x \in X$ such that $f(x) \neq g(x)$. Let $h: Y \longrightarrow \mathbb{C}$ be some function which is zero at $f(x)$ and 1 at $g(x)$. (Such a function exists since Y is normal.) Then $\mathbb{C}(f)(h) \neq \mathbb{C}(g)(h)$ since these two maps disagree at x.

////.

1.3 This is not a full embedding as can be seen by taking $X = Y =$ point. There are precisely two (ring) automorphisms of $\mathbb{C}$: the identity and conjugation. Only the identity is induced by a continuous map. We show in Th. 1.8 that the image of the embedding comprises one half of all the ring homomorphisms, and the other half comprises all the conjugates.

1.4 Conjugation $\mathbb{C} \longrightarrow \mathbb{C}$ induces a natural transformation

$$\varkappa: \mathbb{C}(X) \longrightarrow \mathbb{C}(X).$$

We denote the image of $f \in \mathbb{C}(X)$ under $\varkappa$ by $\bar{f}$, that is

$$\bar{f}(x) = \varkappa(f)(x) = \overline{f(x)}.$$

1.5 For $x \in X$, let $e_x: \mathbb{C}(X) \longrightarrow \mathbb{C}$ be the evaluation map at x, i.e.

$$e_x(f) = f(x).$$

Since e_x is a surjective ring homomorphism, the kernel $\underline{m}_x$ (which consists of all functions which vanish at x) is a maximal ideal. In fact these are the only maximal ideals, as the following well known proposition shows.

1.6 Proposition

There is a 1-1 correspondence between the points of X and the maximal ideals of $\mathbb{C}(X)$ given by the function

$$x \longmapsto \underline{m}_x$$

Proof: By Urysohn's Lemma, this function is injective. If $\underline{m}$ is a maximal ideal, then we show that there is an $x \in X$ such that every $f \in \underline{m}$ vanishes at x, i.e. $\underline{m} \subseteq \underline{m}_x$ whence $\underline{m} = \underline{m}_x$ by maximality.

For, assume for a contradiction that such an x does not exist. Then for each $x \in X$, there is a function $f_x \in \underline{m}$ such that $f_x(x) \neq 0$. Since $\underline{m}$ is an ideal, we can assume that $f_x(x) = 1$. Let U_x be the set of points in X

$$U_x = \left\{ x' \in X : |f_x(x')| > \tfrac{1}{2} \right\}$$

The sets U_x form an open cover of X, so by compactness a finite number say $U_{x_1}, \ldots, U_{x_n}$ cover X. Since $f_x \in \underline{m}$ and $\bar{f}_x \in \mathbb{C}(X)$, $f_x \bar{f}_x \in \underline{m}$ so the element

$$f_{x_1}\bar{f}_{x_1} + \cdots + f_{x_n}\bar{f}_{x_n} \quad \text{is in } \underline{m}$$

and does not vanish anywhere, so is a unit. But maximal ideals do not contain units which is a contradiction.

////.

1.7 If G is a group we can associate to it a category $\underline{G}$ in a natural way as follows. There is only one object in $\underline{G}$, and its morphisms are the elements of G with composition of morphisms being the multiplication in the group. Thus if $\underline{M}$ is a category, we can consider the category $\underline{M} \times \underline{G}$.

1.8 Theorem

Let $\underline{B} \subset \underline{R}$ be the full subcategory generated by $\underline{C}_{con}$. Then

$$\underline{B} \cong \underline{C}_{con} \times \underline{\mathbb{Z}_2}$$

Proof: We think of $\mathbb{Z}_2$ as being the set $(1, \varkappa)$ where $\varkappa^2 = 1$. The objects of $\underline{B}$ are rings of the form $\mathbb{C}(X)$ for some (particular) X and whose morphisms are ring homomorphisms.

Define $F:\underline{C}_{con} \times \mathbb{Z}_2 \longrightarrow \underline{B}$ by

$$F(X) = \mathbb{C}(X) \quad \text{for } X \in \text{Ob } \underline{C}_{con}$$

If $(f,a):X \longrightarrow Y$ in $\underline{C}_{con} \times \mathbb{Z}_2$ then

$$F(f,a):\mathbb{C}(Y) \longrightarrow \mathbb{C}(X) \text{ is}$$

$$F(f,a) = f^* = \mathbb{C}(f) \quad \text{if } a = 1 \in \mathbb{Z}_2$$

$$= f^*\varkappa = \mathbb{C}(f)\varkappa \quad \text{if } a = \varkappa \in \mathbb{Z}_2$$

Clearly F is a functor. In order to define the inverse functor $G:\underline{B} \longrightarrow \underline{C}_{con} \times \mathbb{Z}_2$ we have to examine a ring homomorphism

$$\varphi:\mathbb{C}(X) \longrightarrow \mathbb{C}(Y)$$

Let $i:X \longrightarrow \mathbb{C}$ be the constant map to the complex number i. Since $i^2 = -1$, we have $[\varphi(i)]^2 = -1$ and since Y is connected we deduce that either $\varphi(i) = i$ or $\varphi(i) = -i = \bar{i}$: let $sn(\varphi) = 1$ or $\varkappa \in \mathbb{Z}_2$ accordingly.

Now suppose $y \in Y$. Then $e_y\varphi:\mathbb{C}(X) \longrightarrow \mathbb{C}$ is a ring homomorphism so the kernel is a maximal ideal in $\mathbb{C}(X)$. By Prop.1.6 there is a unique point $x \in X$ such that this ideal is $\underline{m}_x$. Define $f(y) = x$. This defines a function $f:Y \longrightarrow X$. Now the two ring homomorphisms e_x and $e_y\varphi$ from $\mathbb{C}(X)$ to $\mathbb{C}$ are both zero on $\underline{m}_x$ so both induce ring isomorphisms

$$\frac{\mathbb{C}(X)}{\underline{m}_x} = \mathbb{C} \longrightarrow \mathbb{C}$$

Since there are only two ring isomorphisms from $\mathbb{C}$ to $\mathbb{C}$ we deduce that either $e_x = e_y\varphi$ or $e_x\varkappa = e_y\varphi$. We now return to the 2 possibilities for $sn(\varphi)$

<u>Case 1</u>. $sn(\varphi) = 1$. Then for any $y \in Y$,

$$e_y\varphi(i) = e_y(i) = i = e_x(i)$$

so for all $y \in Y$ we have $e_y\varphi = e_{f(y)}$.

<u>Case 2</u>. $sn(\varphi) = \varkappa$. Then for any $y \in Y$,

$$e_y\varphi(i) = e_y(-i) = -i = e_x'$$

so for all $y \in Y$ we have $e_y\varphi = e_{f(y)}\varkappa$.

Now because X,Y are compact hausdorff, the function $f:Y \longrightarrow X$

is continuous iff for all $g \in \mathbb{C}(X)$, $gf \in \mathbb{C}(Y)$. So suppose $g \in \mathbb{C}(X)$. Then if $sn(\varphi) = 1$, $gf = \varphi(g) \in \mathbb{C}(Y)$. If $sn(\varphi) = \varkappa$, $gf = \varphi(g)\varkappa \in \mathbb{C}(Y)$. So in either case f is continuous, so we can define

$$G(\varphi): X \longrightarrow X \text{ in } \underline{C}_{con} \times \underline{\mathbb{Z}}_2 \quad \text{by}$$
$$G(\varphi) = (f, sn(\varphi)).$$

It is easy to see thet G is a functor, and that GF and FG are each naturally equivalent to the identity.

////.

<u>1.9</u> For $X \in \mathrm{Ob}\, \underline{C}$, let Vect(X) be the category of (finite dimensional) complex vector bundles over X. A map $f: X \longrightarrow Y$ in $\underline{C}$ induces a functor $f^*: \mathrm{Vect}(Y) \longrightarrow \mathrm{Vect}(X)$ whcih associates to the bundle E over Y the induced bundle f^*E over X. We quote an important theorem due to Swan (6).

<u>1.10 Theorem</u>

<u>There is an equivalence of categories</u>

$$\Gamma_X : \mathrm{Vect}(X) \longrightarrow P(\mathbb{C}(X))$$

<u>Moreover, this is natural in the sense that if $f: X \longrightarrow Y$ in $\underline{C}$ then</u>

$$\mathrm{Vect}(Y) \xrightarrow[f^*]{} \mathrm{Vect}(X) \xrightarrow[\Gamma_X]{} P(\mathbb{C}(X))$$

<u>and</u>

$$\mathrm{Vect}(Y) \xrightarrow[\Gamma_Y]{} P(\mathbb{C}(Y)) \xrightarrow[f^*]{} P(\mathbb{C}(X))$$

<u>are naturally equivalent functors</u>. ////.

The functor Γ_X associates to the vector bundle E the module of sections $\Gamma_X(E)$. The conjugation map $\varkappa: \mathbb{C}(X) \longrightarrow \mathbb{C}(X)$ induces a functor $\varkappa: P(\mathbb{C}(X)) \longrightarrow P(\mathbb{C}(X))$ which takes $\Gamma_X(E)$ to $\mathbb{C}(X) \otimes_{\varkappa} \Gamma_X(E) \cong \Gamma_X(\bar{E})$, where $\bar{E}$ denotes the conjugate bundle. But $\bar{E} \cong E^{\vee}$, where $E^{\vee}$ denotes the dual bundle (due to properties of the inclusion $U_n(\mathbb{C}) \hookrightarrow GL_n(\mathbb{C})$), and $\Gamma_X(E^{\vee}) \cong (\Gamma_X(E))^{\vee} = \mathrm{Hom}_{\mathbb{C}(X)}(\Gamma_X(E), \mathbb{C}(X))$.
Hence we conclude:

1.11 Proposition

If $\varkappa:\mathbb{C}(X) \longrightarrow \mathbb{C}(X)$ induces $\varkappa:\underline{P(\mathbb{C}(X))} \longrightarrow \underline{P(\mathbb{C}(X))}$, and $P \in \mathrm{Ob}\ \underline{P(\mathbb{C}(X))}$, then $\varkappa(P) \cong P$. ////.

1.11.1 Corollary

If $\varphi:\mathbb{C}(X) \longrightarrow \mathbb{C}(Y)$ in $\underline{R}$, and $G(\varphi) = (f, \mathrm{sn}\,\varphi)$ (see 1.8) and $E \in \mathrm{Ob}\ \underline{\mathrm{Vect}(X)}$ then

$$\begin{aligned}\varphi_* \Gamma'_X(E) &\cong \Gamma_X(f^*E) \quad \text{if } \mathrm{sn}\,\varphi = 1\\ &\cong \Gamma_X(f^*E^\vee) \quad \text{if } \mathrm{sn}\,\varphi = \varkappa.\end{aligned}$$

////

1.12 An important consequence of Theorem 1.10 is the link between topological K-theory, and algebraic K-theory.

Let $K_t:\underline{C} \longrightarrow \underline{R}$ be topological K-theory (defined in terms of complex vector bundles), and let $H^0_t:\underline{C} \longrightarrow \underline{R}$ be the functor which associates to a compact space X, the ring of continuous integer-valued functions on X. Let $r_t:K_t \longrightarrow H^0_t$ be the natural transformation induced by the function which assocaates to the vector bundle E over X the function $x \longmapsto \dim E_x$, and let $i_t:H^0_t \longrightarrow K_t$ be the natural transformation induced by the function which associates to a functio n f on X the trivial vector bundle whose fibre dimension at x is f(x).

1.13 Proposition

(i) $K_t(X)$ and $H^0_t(X)$ are normal λ-rings.

(ii) the maps r_t and i_t are λ-maps, and $r_t i_t = 1$. ////.

(This is the well known topological version of Theorem 0.17.)

1.14 By virtue of 1.10, there is a natural equivalence .

$$\alpha : K_t(X) \longrightarrow K(\mathbb{C}(X))$$

which is in fact a λ-ring isomorphism. The rest of this section is devoted to the proof of the following theorem:

1.15 Theorem:

There is a natural equivalence

$$\beta : H^0_t(X) \longrightarrow H^0(\mathbb{C}(X))$$

such that the following diagrams commute:

$$\begin{tikzcd} K_t(X) \arrow[r, "r_t"'] \arrow[d, "\alpha"'] & H^0_t(X) \arrow[d, "\beta"] \\ K(\mathbb{C}(X)) \arrow[r, "r"'] & H^0(\mathbb{C}(X)) \end{tikzcd} \qquad \begin{tikzcd} H^0_t(X) \arrow[r, "i_t"'] \arrow[d, "\beta"'] & K_t(X) \arrow[d, "\alpha"] \\ H((C(X)) \arrow[r, "i"'] & K(C(X)) \end{tikzcd}$$

Before we can prove this theorem, we need some technical lemmas.

1.16 Lemma *If* $\mathfrak{p}_1, \mathfrak{p}_2 \in$ spec A, *and* $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ *and if* $P \in$ Ob *P(A)* *then*

$$r_P(\mathfrak{p}_1) = r_P(\mathfrak{p}_2)$$

Proof: The inclusion $A \setminus \mathfrak{p}_2 \subseteq A \setminus \mathfrak{p}_1$ induces a ring homomorphism $A_{\mathfrak{p}_2} \longrightarrow A_{\mathfrak{p}_1}$ such that the diagram

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ A_{\mathfrak{p}_2} \arrow[rr] & & A_{\mathfrak{p}_1} \end{tikzcd}$$

commutes.

Now
$$\begin{aligned} P_{\mathfrak{p}_1} = A_{\mathfrak{p}_1} \otimes_A P &= A_{\mathfrak{p}_1} \otimes_{A_{\mathfrak{p}_2}} (A_{\mathfrak{p}_2} \otimes_A P) \\ &= A_{\mathfrak{p}_1} \otimes_{A_{\mathfrak{p}_2}} (A_{\mathfrak{p}_2})^{r_P(\mathfrak{p}_2)} \\ &= (A_{\mathfrak{p}_1})^{r_P(\mathfrak{p}_2)} \\ &= (A_{\mathfrak{p}_1})^{r_P(\mathfrak{p}_1)} \end{aligned}$$

whence $r_P(\mathfrak{p}_1) = r_P(\mathfrak{p}_2)$.

////.

1.17 Corollary. r_P *is determined by its values on the maximal ideals.*

////.

1.18 If $\underline{m} \subset A$ is a maximal ideal, then $A/\underline{m}$ is a field. If $P \in \underline{P}(A)$ then $A/\underline{m} \otimes_A P \in \mathrm{Ob}\ \underline{P}(A/m)$ is thus a finite dimensional vector space over $A/\underline{m}$.

1.19 Lemma If $\underline{m} \subset A$ is a maximal ideal, and $P \in \underline{P}(A)$ then

$$r_P(\underline{m}) = \dim_{A/\underline{m}} (A/\underline{m} \otimes_A P)$$

Proof: The map $A \longrightarrow A_{\underline{m}}$ takes $\underline{m} \subset A$ to the unique maximal ideal say $\hat{\underline{m}} \subset A_{\underline{m}}$, and so induces a map $A/\underline{m} \longrightarrow A_{\underline{m}}/\hat{\underline{m}}$ which is an isomorphism, and moreover the following diagram of natural maps commutes:

$$\begin{array}{ccc} A & \longrightarrow & A_{\underline{m}} \\ \downarrow & & \downarrow \\ A/\underline{m} & \xrightarrow{\cong} & A_{\underline{m}}/\hat{\underline{m}} \end{array}$$

Now $P_{\underline{m}} = A_{\underline{m}} \otimes_A P = (A_{\underline{m}})^{r_P(\underline{m})}$.

But $A/\underline{m} \otimes_A P = A_{\underline{m}}/\hat{\underline{m}} \otimes_A P = A_{\underline{m}}/\hat{\underline{m}} \otimes_{A_{\underline{m}}} (A_{\underline{m}} \otimes_A P)$

$$= A_{\underline{m}}/\hat{\underline{m}} \otimes_{A_{\underline{m}}} (A_{\underline{m}})^{r_P(\underline{m})}$$

$$= (A_{\underline{m}}/\hat{\underline{m}})^{r_P(\underline{m})}$$

$$= (A/\underline{m})^{r_P(\underline{m})}$$

$$\text{so } \dim_{A/\underline{m}}(A/\underline{m} \otimes_A P) = r_P(\underline{m}).$$

////.

1.20 For the rest of this section, let $X \in \mathrm{Ob}\ \underline{C}$ and let $R = C(X)$. By Prop.1.6, there is a unique maximal ideal corresponding to each point of X, and moreover these are the only maximal ideals. Let $\underline{x}$ denote the maximal ideal corresponding to the point $x \in X$, i.e.

$$\underline{x} = \{ f \in R: f(x) = 0.\}$$

Let $e_x : R \longrightarrow \mathbb{C}$ be the evaluation map $e_x(g) = g(x)$. This is clearly surjective with kernel $\underline{x}$. Let $E \in \mathrm{Ob}\ \underline{\mathrm{Vect}\ (X)}$, and let $\Gamma(E)$ denote the corresponding element of $\underline{P(R)}$.

1.21 Lemma. $E_x \cong \Gamma(E)_{\underline{x}}$ as $\mathbb{C}$-vector spaces, where E_x denotes the fibre over the point x.

Proof: Define $a: R/\underline{x} \otimes_R \Gamma(E) \longrightarrow E_x$ by $z \otimes s \longmapsto zs(x)$, where we identify $R/\underline{x}$ with $\mathbb{C}$ by virtue of 1.20. Clearly a is a vector space homomorphism between finite dimensional vector spaces. To prove it is an isomorphism therefor, it suffices to show it is surjective. If $v \in E_x$, then choose some sectiob $s \in \Gamma(E)$ such that $s(x) = v$. Then $v = a(1 \otimes s)$.

////.

1.22 COrollary If $\underline{p} \in$ spec R, and $x \in X$ is a common zero of p, then $r_p(p) = \dim E_x$ (as a $\mathbb{C}$-vector space.)

Proof: x is a common zero of $\underline{p}$ iff $\underline{p} \subseteq \underline{x}$. By Lemma 1.16 $r_p(\underline{p}) = r_p(\underline{x})$. By Prop.1.19, and Lemma 1.21, $r_p(x) = \dim E_x$.

////.

1.23 We now define the map $\beta: H_t^0(X) \longrightarrow H^0(\mathbb{Z}(X))$. An element of $H_t^0(X)$ is a continuous map $f: X \longrightarrow \mathbb{Z}$, i.e. a function which associates an integer to each component of X.

Suppose $\underline{p} \in$ spec R. Then the set of common zeros of $\underline{p}$ is contained in a unique component of X. [For if x,y are common zeros which l e in different components, say U,V then let $u: X \longrightarrow \mathbb{C}$ be the function which is 1 on U and 0 elsewhere, and let $v: X \longrightarrow \mathbb{C}$ be the function which is 1 ou V and zero elsewhere. Then $u,v \notin \underline{p}$ since they do not vanish at both x and y, but $uv = 0 \in \underline{p}$, contradicting the fact that $\underline{p}$ is prime. Choose an $x \in X$ such that $\underline{p} \subseteq \underline{x}$, and define $\beta(f):$ spec $R \longrightarrow \mathbb{Z}$ by $\beta(f)(\underline{p}) = f(x)$. Then $\beta(f)$ is continuous, and so defines a map $H_t^0(X) \longrightarrow H^0(\mathbb{C}(X))$ which is a λ-map. To show this is an equivalence, we construct an inverse.

Suppose $g:$ spec $R \longrightarrow \mathbb{Z}$ is continuous. Let $U_n = g^{-1}\{n\}$, which is an open and closed subspace. By Prop.0.9, there is a unique idempotent $e_n \in R$ such that $U_n = W(Re_n)$. Now $e_n: X \longrightarrow \mathbb{C}$ being an idempotent is either 0 or 1 on each component of X. Let

$$X_n = \{ x \in X : e_n(x) = 1 \} \text{ - a union of components.}$$

Since $n \neq m$ implies $e_n e_m = 0$, $X_n \cap X_m = \emptyset$ for $n \neq m$, and since the U_n cover spec R, the X_n cover X. Define $\nu(g): X \longrightarrow \mathbb{Z}$ to take the value n on X_n. This construvtion defines a map $H^0(\mathbb{Z}(X)) \longrightarrow H^0_t(X)$ which is clearly an inverse for β.

The proof of Theorem 1.15 is now an easy consequence of Cor.1.22.

2. Partitions

It will be useful in the subsequent sections to have an alternative description of $H^0(A)$ in terms of partitions, which we describe now.

2.1 Definition If A is a commutative ring with identity, then a block of A is an ideal which, with addition and multiplication induced by that of A, is a commutative ring with identity.

We do not require that the identity of the block , is the identity of A, although it is an idempotent of A. Clea ·y there is a 1-1 correspondence between blocks and idempotents. For example, the zero ideal is a block.

2.2 If B_1 and B_2 are blocks in A, then the ideal B_1B_2 consisting of finite sums of elements of the form xy with $x \in B_1$ and $y \in B_2$ is also a block, which we denote by $B_1 \otimes B_2$.

De . 2.2.1 Two blocks B_1, B_2 are orthogonal if $B_1 \otimes B_2 = 0$

2.2.2 If 2 blocks B_1, B_2 are orthogonal, then the sum of the 2 ideals is a block, denoted by $B_1 \oplus B_2$. More generally, if $B_1, \ldots, B_n$ is a set of pairwise orthogonal blocks, then the ideal $B_1 + \ldots + B_n$ is a block, denoted by $B_1 \oplus \ldots \oplus B_n$, ot by $\bigoplus_{i=1}^{n} B_i$.

2.3 Let S(A) be the set of sequences $(X_0, X_1, \ldots, X_n, \ldots)$ of blocks of A such that

(i) only a finite number of block X_i are non-zero

(ii) the maps $A \longrightarrow \prod_{i \geq 0} X_i \qquad (a \longmapsto \{ae_i\})$

$$\prod_{i \geq 0} X_i \longrightarrow A \qquad (\{x_i\} \longmapsto \sum_{i \geq 0}' x_i)$$

are inverse (ring) isomorphisms.

2.3.1 It is a cosequence of (ii) that the blocks X_i are pairwise orthogonal.

2.3.2 We distinguish 2 special elements of $S(A)$ namely

$$0 = (A,0,0,\ldots,0,\ldots)$$

and $\quad 1 = (0,A,0,\ldots,0,\ldots)$.

2.4 If $X = (X_0, X_1, \ldots)$ and $Y = (Y_0, Y_1, \ldots)$ are in $S(A)$, consider the sequence denoted by $X \oplus Y$, where

$$(X \oplus Y)_n = \bigoplus_{i+j=n} (X_i \otimes Y)$$

(Since by 2.3.1 the blocks X_i are pairwise orthogonal, the same is true for the blocks $X_i \otimes Y_j$, and so $(X \oplus Y)_n$ is again a block.)

2.5 Proposition The sequence $X \oplus Y$ is in $S(A)$. The operation '$\oplus$' defines on $S(A)$ the structure of a semigroup, with zero element 0 (see 2.3.2).

Proof: (i) If $X_i = 0$ for $i \geq n$, and $Y_j = 0$ for $j \geq m$, then $(X \oplus Y)_k = 0$ for $k \geq n+m$, so only a finite number of $(X \oplus Y)_k$ are non-zero.

(ii) Condition (ii) of 2.3 is equivalent to the condition that if e_i is the identity in X_i, then $\sum' e_i = 1$, and $e_i e_j = 0$ for $i \neq j$. Let f_j be the identity element in Y_j. Then the identity element in $(X \oplus Y)_n$ is $\sum'_{i+j=k} e_i f_j$. Then

$$\sum_k{}' \sum_{i+j=k}{}' e_i f_j = (\sum_i{}' e_i)(\sum_j{}' f_j) = 1$$

and $$(\sum_{i+j=k}{}' e_i f_j)(\sum_{r+s=k'}{}' e_r f_s) = \sum_{i+j=k}{}' \sum_{r+s=k'}{}' e_i e_r f_j f_s$$

$$= 0 \text{ if } k \neq k' \text{ since then either}$$

$i \neq r$ or $j \neq s$.

Thus $X \oplus Y \in S(A)$. The rest of the proposition is trivial. ////.

2.6 For example $(1 \oplus 1)_k = \bigoplus_{i+j=k} (1_i \otimes 1_j)$

$$= 0 \text{ if } k \neq 2$$

$$= A \text{ if } k = 2.$$

Inductively we dedu e that

$$\underset{n\text{-times}}{1 \oplus 1 \oplus \ldots \oplus 1} = (0,0,\ldots,0,A,0,\ldots.)$$

where the A occurs in the n-th term.

2.7 In fact it is psosible to define a multiplication on $S(A)$ which gives it the structure o. a semiring, with identity 1 (see 2.3.2). For if $X,Y \in S(A)$, the sequence

$$(X \otimes Y)_n = \bigoplus_{ij=n} (X_i \otimes Y_j)$$

is again in $S(A)$, by a similar argument to Prop. .5. To prove that multiplication is distributive, let $X,Y,Z \in S(A)$. Then

$$((X \otimes Y) \oplus (X \otimes Z))_n = \bigoplus_{i+j=n} \bigoplus_{\substack{rs=i \\ pq=j}} (X_r \otimes Y_s \otimes X_p \otimes Z_q)$$

$$= \bigoplus_{i+j=n} \bigoplus_{\substack{rs=i \\ rq=i}} (X_r \otimes Y_s \otimes Z_q)$$

$$=(X \otimes (Y \oplus Z))_n .$$

2.8 If $f:A \longrightarrow B$ is a map in R it induces a semiring homomorphism $S(f):S(A) \longrightarrow S(B)$ as follows. If $X \in S(A)$ and e_i is the identity of X_i, then $\sum e_i = 1$ and $e_i e_j = 0$ for $i \neq j$. Hence $\sum f(e_i) = f(1) = 1$ and $f(e_i)f(e_j) = 0$ for $i \neq j$. Let Y_i be the ideal in B generated by the idempotent $f(e_i)$. Then Y_i is a block, and $Y = (Y_0,Y_1,\ldots) \in S(B)$. Define $S(f)(X) = Y$.

Thus S is a functor from R to the category of (commutative) semirings.

2.9 Suppose $X \in S(A)$. Then if e_i is the identity of X_i, we know that $X_i = Ae_i$. By Prop.0.9 the subset $W(X_i) = W(Ae_i)$ is open and closed in spec A. Define $\emptyset(X)$:spec $A \longrightarrow \mathbb{Z}$ to take the value i on the subspace W_i. Then $\emptyset(X)$ is continuous.

2.10 Proposition. The function $\emptyset : S(A) \longrightarrow \{$continuous maps from spec A to the non-negative inetegrs$\}$ is a natural equivalence of semirings.

Proof: We construct an inverse for $\emptyset$. If $f:\text{spec } A \longrightarrow \mathbb{Z}$ takes non-negative values and is continuous, let $U_i = f^{-1}\{i\}$. By Prop.0.9 there is a unique idempotent e_i such that $U_i = W(Ae_i)$. Let $X_i = Ae_i$. Then X_i is a block, and only a finite number are non-zero. Since $U_i \cap U_j = \emptyset$ for $i \neq j$, the blocks X_i and X_j are orthogonal, and since the U_i cover spec A, the map $\prod_{i \geq 0} X_i \longrightarrow A$ is an isomorphism. Thus the sequence $(X_0, X_1, \ldots) \in S(A)$ and this construction defines the inverse for $\emptyset$.

////.

2.11 Corollary Let $L(A)$ denote the Grothendieck ring of $S(A)$. Then $\emptyset$ induces an isomorphism $\emptyset : L(A) \longrightarrow H^0(A)$. ////.

2.12 If $P \in P(A)$, then $r_P : \text{spec } A \longrightarrow \mathbb{Z}$ takes non-negative values, so by 2.10 determines an elemnnt $X(P) \in S(A)$, called the partition (of A) induced by P. We note that $X(P \oplus Q) = X(P) \oplus X(Q)$ and $X(P \otimes Q) = X(P) \otimes X(Q)$.

The natural map $A \longrightarrow \prod_{i \geq 0} X(P)_i$ induces commutative diagrams

$$\begin{tikzcd} K(A) \arrow[r, "r"] \arrow[d] & H^0(A) \arrow[d] \\ \prod_i K(X(P)_i) \arrow[r, "r"] & \prod_i H^0(X(P)_i) \end{tikzcd}$$

and

$$\begin{tikzcd} K(A) \arrow[d] & H^0(A) \arrow[l, "i"'] \arrow[d] \\ \prod_i K(X(P)_i) & \prod_i H^0(X(P)_i) \arrow[l, "i"'] \end{tikzcd}$$

where by Theorem 0.19, the vertical maps are isomorphisms of λ-rings.

In particular, we notice the the module $P_i = X(P)_i \otimes_A P$ (which is the image of P in $K(X(P)_i)$ has constant rank equal to i. This then gives us an important principle which enables us to deduce results about arbitrary projective modules from results about projective modules of constant rank. This principle will be used in later sections.

3. Operations in Algebraic K-theory.

3.1 Given two functors F,G from a category $\underline{A}$ to a category $\underline{B}$, let Op(F,G) denote the set of natural transformations from F to G. If $\underline{B}$ has some algebraic structure, we can give Op(F,G) this structure in a 'pointwise' manner. For example, if $\underline{B}$ is the category of abelian groups and $a,b \in Op(F,G)$ then we can define a + b and -a by the rules

$$(a + b)(X) = a(X) + b(X)$$

$$(-a)(X) = -a(X) \qquad \text{for } X \in \text{Ob } A.$$

3.2 For $A \in \text{Ob } \underline{R}$, let $Rk_0(A) = Ker(r:K(A) \longrightarrow H^0(A))$. Then $Rk_0(A)$ is an ideal in K(A) and by virtue of Theorem 1.7 (iv) there is a natural equivalence $K(A) \cong Rk_0(A) \oplus H^0(A)$.

3.3 In this section we are interested in Op(K,K). By the generalities of 3.1, this is a commutative ring with identity. The isomorphism of 3.2 induces a decomposition

3.3.1 $$Op(K,K) \cong Op(Rk_0,K) \oplus Op(H^0,Rk_0) \oplus Op(H^0,H^0)$$

3.4 Proposition

(i) $Op(H^0,Rk_0) = 0$

(ii) $Op(H^0,H^0) = \mathbb{Z}^{\mathbb{Z}}$

Proof. (i). Suppose not. Then there is an operation $\eta : H^0 \longrightarrow Rk_0$ and an $A \in \text{Ob } \underline{R}$ such that $\eta(A):H^0(A) \longrightarrow Rk_0(A)$ is non zero, i.e. there is an $f \in H^0(A)$ such that $\eta(A)(f) \neq 0$.

Case (a). If $f:\text{spec } A \longrightarrow \mathbb{Z}$ takes a constant value, say n, let $f':\text{spec } \mathbb{Z} \longrightarrow \mathbb{Z}$ take constant value n. Let $i:\mathbb{Z} \longrightarrow A$ be the natural map. Then the following diagram commutes:

$$\begin{tikzcd} H^0(A) \arrow[r, "\eta(A)"] & Rk_0(A) \\ H^0(\mathbb{Z}) \arrow[u, "i_*"] \arrow[r, "\eta(\mathbb{Z})"'] & Rk_0(\mathbb{Z}) \arrow[u, "i_*"'] \end{tikzcd}$$

But $i_*(f') = f$, and $Rk_0(\mathbb{Z}) = 0$, whence $\eta(A)(f) = 0$.

Case (b). Suppose f does not take constant values. Then by the methods of section 2, f induces a partition of A into subrings on which f takes constant values. We then apply case (a) to each subring.

(ii) An operation $\theta : H^0 \longrightarrow H^0$ induces a function

$$\theta(\mathbb{Z}) : H^0(\mathbb{Z}) \longrightarrow H^0(\mathbb{Z}) \qquad \text{i.e. } \mathbb{Z} \longrightarrow \mathbb{Z}$$

This defines a function $\alpha : Op(H^0, H^0) \longrightarrow \mathbb{Z}^{\mathbb{Z}}$. Conversely, giveb a function $g : \mathbb{Z} \longrightarrow \mathbb{Z}$, it induces an operation $H^0 \longrightarrow H^0$ by composition This defines a function $\beta : \mathbb{Z}^{\mathbb{Z}} \longrightarrow Op(H^0, H^0)$ such that $\alpha\beta = 1$. It now suffices to prove thet α is injective. The proof of this is analogous to (i) above.

////.

3.5 Corollary

$$Op(K,K) = Op(Rk_0, K) \oplus \mathbb{Z}^{\mathbb{Z}}$$

////.

4. Dimension properties.

For $P \in Ob\ \underline{P(A)}$ we define 4 integers:

4.1 Definition

(i) $m(P) = \min \{ r_P(\underline{p}) : \underline{p} \in \operatorname{spec} A \}$

(ii) $M(P) = \max \{ r_P(\underline{p}) : \underline{p} \quad \operatorname{spec} A \}$

(iii) $l(P) = \max \{ n : \exists\, Q \in Ob\ P(A) \text{ such that } Q \oplus A^n = P \}$

(iv) $g(P) = \min \{ n : \exists\, Q \in Ob\ P(A) \text{ such that } Q \oplus P = A^n \}$

4.2 We note that $l(P)$ is the maximum number of inearly independent elements spanning a direct summand, and $g(P)$ is the minimum number of generators.

Lemma

4.3 ~~Proposition~~:

(i) If $P \in \underline{P(A)}$ then $l(P) \leqslant m(P) \leqslant M(P) \leqslant g(P)$.

(ii) and $\lambda^n(P) = 0$ for $n > M(P)$

Proof: By definition, there is a $Q \in P(A)$ such that $Q \oplus A^{l(P)} = P$

If $\underline{p} \in$ spec A, then $r_P(\underline{p}) = r_Q(\underline{p}) + l(P)$ and $r_Q(\underline{p}) \geqslant 0$. Hence $l(P) \leqslant m(P)$. Similarly $M(P) \geqslant g(P)$, and clearly $m(P) \leqslant M(P)$.

(ii) If $n > M(P)$, and $\underline{p} \in$ spec A, then $\lambda^n(P)_{\underline{p}} = \lambda^n(P_{\underline{p}}) = 0$ since $n > M(P) \geqslant r_P(\underline{p})$. Hence $\lambda^n(P)$ is locally zero, so by Prop.0.7 (i), $\lambda^n(P) = 0$.

////.

4.4 If $x \in K(A)$, then $\gamma_t(x) \in 1 + K(A)[[t]]^+$ has an inverse, which we denote by $\bar{\gamma}_t(x) = 1 + \sum_{n \geqslant 1} \bar{\gamma}^n(x)t^n$.

4.5 Proposition If $P \in \underline{P(A)}$

(i) $\gamma^i[P] = 0$ for $i > M(P) - l(P)$

(ii) $\bar{\gamma}^i[P] = 0$ for $i > g(P) - m(P)$

Proof: (i) Write $P = l(P) \oplus Q$ where $l(P)$ denotes the free module of rank $l(P)$. Then $[P] = [Q]$ and $M(Q) = M(P) - l(P)$. Hence it suffices to show that $\gamma^i[Q] = 0$ for $i > M(Q)$.

Case (a). If r_Q is constant (and equal to $M(Q)$, then by Prop.4.3(ii) $\lambda_t[Q]$ is a polynomial of degree $\leqslant M(Q)$. Thus

$$\begin{aligned}
\gamma_t\, Q &= \gamma_t(Q - M(Q)) = \gamma_t(Q).\, \gamma_t(-M(Q)) \\
&= \lambda_{t/1-t}(Q).(\gamma_t(1))^{-M(Q)} \\
&= \sum_{i \geqslant 0} \lambda^i(Q) \left(\frac{t}{1-t}\right)^i .\, (1 - t)^{M(Q)} \\
&= \sum_{i \geqslant 0} \lambda^i(Q) t^i (1 - t)^{M(Q) - i}
\end{aligned}$$

which is a polynomial of degree $\leqslant M(Q)$..

Case (b). If r_Q is not constant, then Q induces a partition Of A, say $(X_0, X_1, \ldots.)$ where $Q_i = X_i \otimes_A Q$, if not zero, has constant rank equal to i, and where $X_r = 0$ for $r > M(Q)$.

By 2.12, this partition induces a commutative diagram

$$\begin{tikzcd} K(A) \arrow[r] \arrow[d, dashed] & H^0(A) \arrow[d] \\ \prod K(X_r) \arrow[r] & \prod H^0(X_r) \end{tikzcd}$$

where the vertical maps are λ-ring isomorphisms. Thus $\gamma^i[Q]$ in $K(A)$ maps to $\gamma^i[Q_r] = \gamma^i(Q_r - r)$ in $K(X_r)$, and so by case (a),

$$\gamma^i(Q_r - r) = 0 \text{ for } i > r.$$

Hence $\gamma^i[Q] = 0$ for $i > \max\{r : K(X_r) \neq 0\}$

i.e. for $i > M(Q)$, since $X_r = 0$ for $i > M(Q)$.

(ii) Write $P \oplus Q = g(P)$. Then $[P \oplus Q] = [g(P)] = [0]$, and so

$$\gamma_t[P \oplus Q] = \gamma_t[P] . \gamma_t[Q] = \gamma_t[0] = 1$$

so $\gamma_t(Q) = \bar{\gamma}_t[P]$.

If $\underline{p} \in \operatorname{spec} A$, then $r_P(\underline{p}) + r_Q(\underline{p}) = g(P)$, and so

$$\begin{aligned} M(Q) &= \max\{r_Q(\underline{p}) : \underline{p} \in \operatorname{spec} A\} \\ &= \max\{g(P) - r_P(\underline{p}) : \underline{p} \in \operatorname{spec} A\} \\ &= g(P) - \min\{r_P(\underline{p}) : \underline{p} \in \operatorname{spec} A\} \\ &= g(P) - m(P) \end{aligned}$$

But by (i), $\gamma_t\, Q$ is a polynomial of degree $\leqslant M(Q)$, hence $\bar{\gamma}_t\, P$ is a polynomial of degree $\leqslant g(P) - m(P)$.

////.

<u>4.6</u> <u>Definition</u> An element $x \in K(A)$ is positive (written $x \geqslant 0$) if there is a $P \in \underline{P(A)}$ such that $x = P$.

<u>4.7</u> For $x \in Rk_0(A)$, let

$$S_x = \{Q \in \operatorname{Ob} \underline{P(A)} \text{ such that } Q \cong r_Q \text{ and } x + Q \geqslant 0\}$$

Then S_x is non-empty, since we can write $x = P - r_P$ for some $P \in P(A)$ and then $r_P \in S_x$.

<u>4.8</u> <u>Definition</u> For $x \in Rk_0(A)$, define

$$\dim_\Gamma(x) = \min\{M(Q) : Q \in S_x\}$$

4.9 Proposition For $x \in Rk_0(A)$, $\gamma_t(x)$ is a polynomial of degree $\leqslant \dim_K(x)$.

Proof: By definition, there eixsts a $Q \in S_x$ with $M(Q) = \dim_K(x)$, and a $P \in P(A)$ such that $P = x + Q$ in $K(A)$. Then

$$P - r_P = (x + Q) - (r(x) + r_Q)$$
$$= x \quad \text{since } r(x) = 0, \text{ and } Q = r_Q.$$

By Prop.4.5, $\gamma_t(x) = \gamma_t[P]$ is a polynomial of degree $\leqslant M(P)$. But since $P = x + Q$, we have $r_P = r_Q$, so $M(P) = M(Q) = \dim_K(x)$. Hence $\gamma_t(x)$ is a polynomial of degree $\leqslant \dim_K(x)$.

////.

4.10 Corollary For $x \in Rk_0(A)$, there exists an integer $N(x)$ such that

$$\gamma^{i_1}(x).\gamma^{i_2}(x). \quad .\gamma^{i_r}(x) = 0 \text{ when } \sum i_r \geqslant N(x).$$

Proof: This is a direct consequence of Prop.4.9 and a proposition of Atiyah ((7),Prop.3.1.5 p127). ////.

4.11 Corollary Any element $x \in Rk_0(A)$ is nilpotent.

Proof: This follows from Cor.4.10, since $\gamma^1(x) = x$. ////.

4.12 Corollary If $Rk_0(A)$ is finitely generated, it is nilpotent. ////.

4.13 Let $\mathbb{Z}[[s_1, s_2, \ldots]]$ be the ring of integer power series in a countable family of indeterminates (see 0.20 et.seq.). Then if we give each s_i degree 1, we can think of a power series as a formal sum $\sum_n f_n$ where f_n is a homogeneous polynomial of degree n.

Define a map $w: \{s_1, s_2, \ldots\} \longrightarrow Op(Rk_0, K)$ by sending s_i to γ^i. This map induces a ring homomorphism from the ring of polynomials in the $\{s_i\}$, but by 4.10 it induces a map from the power series in the $\{s_i\}$ to $Op(Rk_0, K)$, which we denote by

$$w: \mathbb{Z}[[s_1, s_2, \ldots]] \longrightarrow Op(Rk_0, K)$$

The main result of this section is the following:

4.14 Theorem

$$w:\mathbb{Z}[[s_1,s_2,\ldots]] \longrightarrow Op(Rk_0,K)$$

is a split monomorphism, i.e. there is a (ring) homomorphism

$$v:Op(Rk_0,K) \longrightarrow \mathbb{Z}[[s_1,s_2,\ldots]]$$

such that $vw = 1$.

Proof: Let $Op_{\underline{C}}(Rk_0,K)$ be the operations between Rk_0 and K restricted to the subcategory $\underline{C}$ of $\underline{R}$. There is the a restriction homomorphism

$$u:Op(Rk_0,K) \longrightarrow Op_{\underline{C}}(Rk_0,K).$$

Let $x = uw:\mathbb{Z}[[s_1,s_2,\ldots]] \longrightarrow Op_{\underline{C}}(Rk_0,K)$. Then by Theorem 1.15 and the result of Atiyah ((7),Theorem 3.1.7,page 128), the map x is an isomorphism. Let $v = x^{-1}u$. The result now follows.

////.

In particular, this says that the operations γ^i are linearly independent, and hence non-trivial.

5. The graded ring and relations with the Picard group

5.1 Let $K_n(A)$ be the subgroup of $K(A)$ generated by all monomials of the form

$$\gamma^{i_1}(x_1).\ \gamma^{i_2}(x_2).\ \ \ \gamma^{i_r}(x_r)$$

where $x_j \in Rk_0(A)$ and $\sum i_j \geq n$.

5.2 Proposition (Grothendieck)

(i) $K_0(A) = K(A)$

(ii) $K_1(A) = Rk_0(A)$

(iii) $K_{n+1}(A) \subseteq K_n(A)$

(iv) $K_n(A).K_m(A) \subseteq K_{n+m}(A)$

Proof: (i),(iii) and (iv) are trivial.

(ii) Since $\gamma^1 = 1$, we see $Rk_0(A) \subseteq K_1(A)$. The reverse inclusion also holds since r commutes with the γ^i and $\gamma^i(0) = 0$ for $1 \leq i$.

////.

5.3 We note that this filtration is natural. Let $H^*(A)$ be the associated graded ring. From Prop.5.2 (ii) we see that $H^0(A)$ is well defined.

5.4 For $P \in \mathrm{Ob}\ \underline{P(A)}$, let $c_i(P)$ be the element of $H^i(A)$ determined by $\gamma^i[P] \in K_i(A)$. Then clearly the c_i are natural, and satisfy the Whitney sum formula

$$c_n(P \oplus Q) = \sum_{i+j=n} c_i(P)c_j(Q).$$

Moreover, by Prop.4.5, $c_i(P) = 0$ for $i > M(P)$.

5.5 For $A \in \mathrm{Ob}\ \underline{R}$, let $\mathrm{Pic}(A)$ be the Picard group of A, i.e. the group of (isomorphism classes) of projective modules of rank 1 under tensor product. The rule $P \longmapsto c_1(P)$ defines a function $c_1 : \mathrm{Pic}(A) \longrightarrow H^1(A)$

5.6 Theorem

$$c_1 : \mathrm{Pic}(A) \longrightarrow H^1(A)$$

is an isomorphism of abelian groups.

5.7 e prove this theorem as the result of several lemmas. First we note that this function is a homomorphism. For if $P, Q \in \mathrm{Pic}(A)$ then $(P - 1)(Q - 1) \in K_2(A)$, and so

$$(P \otimes Q - 1) \equiv (P - 1) + (Q - 1) \quad \mathrm{mod}\ K_2(A)$$

i.e. $$c_1(P \otimes Q) = c_1(P) + c_1(Q).$$

5.8 If $P \in \mathrm{Ob}\ \underline{P(A)}$, and P has cons ant rank n, define $\det P = \lambda^n P$ Then det P has rank 1, so belongs to Pic(A). If P is any object in $\underline{P(A)}$, it induces a partition $(X_0, X_1, \ldots)$ of A such that

$$P_i = X_i \otimes_A P$$

(if non-zero) has constant rank (equal to i) as an X_i-mod le. Then $\det P_i$ is an X_i module of rank 1, and we define

$$\det P = \bigoplus_i (A \otimes_{X_i} \det P_i)$$

which is an A- module of rank 1. It is well-known (see Bass, (5) that $\det(P \oplus Q) = \det P . \det Q$, so that this construction induces a homomorphism

$$\det : K(A) \longrightarrow \mathrm{Pic}\,(A)\ .$$

<u>5.9 Proposition</u> <u>If</u> $P, Q \in \mathrm{Ob}\ \underline{P(A)}$ <u>have constant ranks, say p, q, then</u>

(i) $\det(P \otimes Q) = (\det P)^q . (\det Q)^p$

(ii) $\det(\lambda^i P) = (\det P)^n$ <u>where</u> $n = \lambda^{i-1}(p-1)$.

<u>Proof:</u> (i) Let $\otimes^p(P)$ denote the tensor product of p-copies of P. Consider the natural isomorphism

$$\left(\otimes^q(\otimes^p(P))\right) \otimes \left(\otimes^p(\otimes^q(Q))\right) \longrightarrow \otimes^{pq}(P \otimes Q)$$

which 'preserves the order'. This induces a map into $\lambda^{pq}(P \otimes Q)$ which respects the identification

$$\left(\otimes^q(\otimes^p(P))\right) \otimes \left(\otimes^p(\otimes^q(Q))\right) \longrightarrow \left(\otimes^q(\lambda^p(P)\right) \otimes \left(\otimes^p(\lambda^q(Q))\right)$$

and so induces a homomorphism

$$i : \left(\otimes^q(\det P)\right) \otimes \left(\otimes^p(\det Q)\right) \longrightarrow \det P \otimes Q\ .$$

This map is natural, and when P, Q are free is an isomorphism, as can be seen by choosing bases. Hence it is an isomorphism locally, and hence, by Prop.0.7 (ii), it is an isomorphism.

(ii) Let $\lambda^i(p) = m$. Then $mi = p!i/i!(p-i)! = p(p-1)!/(i-1)!(p-i)!$

$$\text{i.e.} \qquad mi = pn.$$

Consider the isomorphism

$$\otimes^n(\otimes^p(P)) \longrightarrow \otimes^m(\otimes^i(P))$$

which 'preserves the order'. This defines a homomorphism

$$\otimes^n(\otimes^p(P)) \longrightarrow \lambda^m(\lambda^i(P))$$

which respects the identification

$$\otimes^n(\otimes^p(P) \longrightarrow \otimes^n(\lambda^p(P))$$

and so induces a homomorphism

$$\otimes^n(\det P) \longrightarrow \det(\lambda^i P)$$

which is an isomorphism for the same reasons as in (i) above.

5.10 Corollary If $x, y \in K(A)$ have constant ranks, say n, m then

(i) $$\det(xy) = (\det x)^m.(\det y)^n$$

(ii) $$\det(\lambda^i x) = (\det x) \quad \text{where} \quad = \lambda^{i-1}(n-1)$$

Proof: (i) Write $x = P - p'$ and $y = Q - q'$ where $P, Q \in \underline{P(A)}$ have constant ranks p, q and where $p - p' = n$, and $q - q' = m$.

Then $xy = (P - p')(Q - q')$, and so

$$\begin{aligned}
\det(xy) &= \det(P \otimes Q).\det(p'q').(\det(P \otimes q'))^{-1}.(\det(Q \otimes p'))^{-1} \\
&= (\det P)^q.(\det Q)^p.(\det P)^{-q'}.(\det Q)^{-p'} \quad \text{since } \det r = 1 \\
&= (\det P)^m.(\det Q)^n \\
&= (\det x)^m.(\det y)^n \quad .
\end{aligned}$$

(ii) $$\begin{aligned}
\det(\lambda^i x) &= \det \lambda^i(P - p') = \det \sum_{r+s=i} \lambda^r P.\,\lambda^s(-p') \\
&= \prod_{r+s=i} \det(\lambda^r P.\,\lambda^s(-p')) \\
&= \prod_{r+s=i} (\det \lambda^r P)^{\lambda^s(-p')} \qquad \text{by (i)} \\
&= \prod_{r+s=i} (\det P)^{\lambda^{r-1}(p-1).\,\lambda^s(-p')} \qquad \text{by Prop. 5.9 (ii)} \\
&= (\det x)^{t'} \quad \text{where } t' = \sum_{r+s=i} \lambda^{r-1}(p-1).\,\lambda^s(-p').
\end{aligned}$$

But $\lambda^{i-1}(n-1) = \lambda^{i-1}(p - 1 - p')$

$$\text{i.e. } t' = \sum_{r+s=i-1} \lambda^r(p-1).\,\lambda^s(-p') = t \quad .$$

////.

5.11 Corollary. $\det|K_2(A) = 0$.

Proof: $K_2(A)$ is generated by monomials of the form

$$\gamma^{i_1}(x_1). \quad .\gamma^{i_r}(x_r)$$

where $x_j \in Rk_0(A)$ and $\sum i_j \geq 2$. Such a monomial is one of two types:

Type (a) - a product of two or more elements of $Rk_0(A)$

Type (b) - $\gamma^i(x)$ where $x \in Rk_0(A)$ and $i \geq 2$.

We show that det is zero on each type.

Type (a) Such an element can be written as a product xy where $x, y \in Rk_0(A)$. Then

$$\det(xy) = (\det x)^{r(y)}.(\det y)^{r(x)}$$
$$= 1, \text{ the zero element in Pic } (A)$$

(by Cor.5.10 (i), since $r(x) = 0 = r(y)$.)

Type (b)

$$\det(\gamma^i(x)) = \det(\lambda^i(x + i - 1))$$
$$= (\det(x + i - 1))^{\lambda^{i-1}(i - 1 - 1)}$$

Now if $i = 2$, the exponent is $\lambda^1(0) = 0$, and if i 2 it is also 0 since $i - 1 > i - 2$.

Thus $\det(\gamma^i(x)) = 1$, the zero element of Pic (A).

Hence det is zero on $K_2(A)$.

////.

5.12 Because of Cor.5.11, the homomorphism $\det : K(A) \longrightarrow \text{Pic } (A)$ factors through a map

$$\widehat{\det} : H^1(A) \longrightarrow \text{Pic } (A).$$

5.13 Proof of Theorem 5.6

We show that c_1 and $\widehat{\det}$ are inverse isomorphisms.

If $P \in \text{Pic } (A)$, then

$$\widehat{\det}.c_1(P) = \det(P - 1) = \det P = P \quad \text{since } r(P) = 1.$$

Thus $\widehat{\det}.c_1 = 1$. To prove the converse, suppose $z \in H^1(A)$ is represented by $P - n$ in $Rk_0(A)$. Then $\widehat{\det}\, z = \det P = \lambda^n P$, so

$$c_1.\widehat{\det}\, z = c_1(\lambda^n P) = (\lambda^n P - 1) \bmod K_2(A).$$

It then suffices to show that

$$(P - n) \equiv (\lambda^n P - 1) \bmod K_2(A).$$

Now by the following lemma (Lemma 5.14), we know that

$$\lambda^n(P) = \sum_{0 \leq i \leq n} \gamma^i(P - n)$$

$$\text{i.e. } \lambda^n P - 1 = P - n + \sum_{2 \leq i \leq n} \gamma^i(P - n)$$

$$\text{whence } \lambda^n P - 1 \equiv P - n \bmod K_2(A).$$

////.

<u>5.14 Lemma</u>

(i) $$\sum_{j=0}^{m} \gamma^j(-m) = 0 \quad \text{for } m > 0.$$

(ii) <u>For</u> $P \in$ <u>$P(A)$, rank n</u> $$\sum_{i=0}^{n} \gamma^i(P-n) = \lambda^n(P)$$

<u>Proof</u>: (i) $$\lambda_t(-m) = (1+t)^{-m}, \text{ and so}$$
$$\gamma_t(-m) = (1-t)^m.$$

Then $\sum_{j=0}^{m} \gamma^j(-m)$ is the result of substituting 't=1' in $\gamma_t(-m)$, which is thus zero.

(ii) $$\sum_{i=0}^{n} \gamma^i(P-n) = \sum_{i=0}^{n} \lambda^i(P-n+i-1)$$
$$= \sum_{i=0}^{n} \sum_{r=0}^{n-r} \lambda^r(P).\lambda^{i-r}(-n+i-1)$$

For $0 \leqslant r \leqslant n$, the coefficient of $\lambda^r(P)$ is
$$\sum_{i=r}^{n} \lambda^{i-r}(-n+i-1) = \sum_{j=0}^{n-r} \lambda^j(-n+r+j-1)$$
$$= \sum_{j=0}^{n-r} \gamma^j(-(n-r))$$

For $r = n$, this is zero by (i). But for $r = n$, this is $\gamma^0(0) = 1$, so
$$\sum_{i=0}^{n} \gamma^i(P-n) = \lambda^n(P).$$

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